Complex Analysis

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Fall 2021

The present notes were made during the course of Complex Analysis MAT214 at the University of Bergen during fall 2021.

The teacher assistant was Arnaud Eychenne, see his personal webpage for the well-corrected exercises.

Contents

1 Definitions and interpretations of complex structures.

1.1 The real case.

1.1.1 Basis.

In this course, we will denote the real vector space of dimension n by $\mathbb{R}^n$, and coordinates by $(x_1, \cdots, x_n)$, with $x_i \in \mathbb{R}$, $\forall i \in [\![1, ; n]\!]$. We will denote the sets without the null vector by $\mathbb{R}* = \mathbb{R}_{\setminus\{0\}}$, and $\mathbb{R}^n_{\setminus\{0\}} = \{(x_1, \cdots, x_n) \neq (0, \cdots, 0)\}$.
Once a field is defined, we define functions on it, such as the exponential, cosine and sine (these two latter functions are real-valued functions, but not defined by the Euler formula! We can define them by series or as unique solutions of ODEs for example).
A natural question occurs: how can you add a point a infinity? What can infinity mean, and what topology can we define on those new sets?

1.1.2 The real projective line.

In $\mathbb{R}^2_{\setminus\{0\}}$, we define the following equivalence relation:

$$v \sim w \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{R}^*, \quad v = \lambda w.$$

Definition 1. *The <u>real projective line</u>, denoted by $\mathbb{P}^1(\mathbb{R})$ or $\mathbb{P}(\mathbb{R})$ is the set $\mathbb{R}^2_{\setminus\{0\}/\sim}$.*

If $(a, b) \in \mathbb{R}^2_{\setminus\{0\}}$, we denote the class of the element by $[a; b]$. In particular, if $b \neq 0$, we have $[a; b] = [\frac{a}{b}; 1] = [\lambda a; \lambda b]$, for any $\lambda \neq 0$. In fact, considering the set $[a, 1]$, with $a \in \mathbb{R}$, we obtain a set of class which is isomorphic to $\mathbb{R}$. Nevertheless only remains one class, or a line in $\mathbb{R}^2_{\setminus\{0\}}$ which was included in the previous definition. This set is $[1, 0]$, and we will consider this set as the point at infinity. We thus obtain:

$$\mathbb{P}^1(\mathbb{R}) \simeq \mathbb{R} \cup \{\infty\}.$$

1.1.3 The real projective plane.

In $\mathbb{R}^3_{\setminus\{0\}}$, we define similarly the equivalence relation: $v \sim w$ if and only if $\exists \lambda \in \mathbb{R}^*$, $v = \lambda w$.

Definition 2. *The <u>real projective plane</u>, denoted by $\mathbb{P}^2(\mathbb{R})$, is defined by $\mathbb{R}^3_{\setminus\{0\}/\sim}$.*

As before, we notice that if $v_z \neq 0$, we get:

$$[v_x, v_y, v_z] = \left[\frac{v_x}{v_z}, \frac{v_y}{v_z}, 1\right].$$

What about infinity? Actually, the set $\{[v_x, v_y, 1], v_x \in \mathbb{R}, v_y \in \mathbb{R}\}$ is isomorphic to $\mathbb{R}^2$. It means that all the lines parallel to the plane $z = 1$ will be at infinity. Furthermore, this set of lines is the one that we studied before! We thus obtain:

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \mathbb{P}^1(\mathbb{R}) = \mathbb{R}^2 \cup \mathbb{R} \cup \{\infty\}.$$

Remark 3. *A notion of dimension is well-defined on those sets, which is $\dim(\mathbb{P}^n(\mathbb{R})) = n$.*

1.2 Definition of $\mathbb{C}$, S^2, $\hat{\mathbb{C}}$ and $\mathbb{P}^1(\mathbb{C})$; geometrical interpretation. [1]

1.2.1 The complex plane.

Goal : define the square root of -1.
A first definition of $\mathbb{C}$ can be done by defining the appropriate laws:

$$(\mathbb{C}, +, \cdot) := (\mathbb{R}^2, +, \star),$$

where the law $\star$ is defined by:

$$(x, y) \star (u, v) := (xu - yv, xv + yu).$$

We denote the point (x, y) in $\mathbb{R}^2$ by $x \cdot 1 + y \cdot i$ in $\mathbb{C}$. By computation: $(0 \cdot 1 + 1 \cdot i) \cdot (0 \cdot 1 + 1 \cdot i) = -1 \cdot 1 + 0 \cdot i$. For short, we also use the following notation $z = x + iy \in \mathbb{C}$, where $x \in \mathbb{R}$, and $y \in \mathbb{R}$.
We define functions from $\mathbb{C}$ to $\mathbb{R}$, such as the usual real part, imaginary part and modulus on this set:

$$\mathrm{Re}(x + iy) = x, \quad \mathrm{Im}(x + iy) = y, \quad |x + iy| = \sqrt{x^2 + y^2}.$$

Another definition of the complex numbers can be derived from the polynomials. Consider the set of real polynomials $\mathbb{R}[X]$, we use the equivalence relation:

$$P \sim Q \quad \Leftrightarrow \quad X^2 + 1 | P - Q.$$

In fact, for any polynomial $P \in \mathbb{R}[X]$, there exists a (unique) polynomial of degree less or equal than 1 which is equivalent to P. This polynomial is the rest by the euclidean division of P by $X^2 + 1$. We get the following isomorphism:

$$\begin{array}{rccc} \phi : & \mathbb{R}[X]_{/\sim} & \to & \mathbb{C} \\ & \overline{aX + b} & \mapsto & b + ia. \end{array}$$

You can check that the usual multiplication on polynomials is sent on the law of complex numbers.
Similarly, we define a subspace of matrices:

$$M := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The following function is an isomorphism of rings:

$$\begin{array}{rccc} \phi : & M & \to & \mathbb{C} \\ & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & \mapsto & a + ib. \end{array}$$

Remark 4. *Why do we need all those definitions? In fact, it leaves the door open for other fields of analysis. For example, other sets can be defined as subspaces of matrices. We can mention the quaternions or the octonions, see for example [5]. Later in the course, we will see that the notions of holomorphy of analyticity of functions are identical. But it does not coincide on any field, such as the quaternions!*

We recall the polar coordinates on $\mathbb{C}$. If $x + iy \in \mathbb{C}$, we define $r := \sqrt{x^2 + y^2}$, and the argument as a real θ such that $x = r\cos(\theta)$, $y = r\sin(\theta)$. We also define the following expression: $e^{i\theta} := \cos(\theta) + i\sin(\theta)$. By geometry or computation, we get:

$$re^{i\theta} \cdot \rho e^{i\phi} = r\rho e^{i(\theta+\phi)}.$$

(Do the computations by yourself to be convinced!)

Definition 5. *We define the* <u>*complex exponential*</u> *e on $\mathbb{C}$ by:*

$$e : \quad x + iy \quad \to e^x e^{iy},$$

where e^x is the real exponential.

Proposition 6.

$$\forall \alpha, \beta \in \mathbb{C}, \quad e^{\alpha+\beta} = e^\alpha e^\beta.$$

It is coherent with the real properties.

To entertain yourself, you can find the solutions of the following equation, for $n \in \mathbb{N}$

$$\omega^n = 6i.$$

Notice that in this case, the complex exponential is definite; take care not to use the logarithm, which is not defined yet!

1..2 Weekly recall of topology

Definition 7. *A collection τ of subsets of a set X is called a topology if:*

1. $\emptyset \in \tau$, $X \in \tau$;
2. *If* $V_i \in \tau$, *for* $i \in [\![1; n]\!]$ *(finite),* $\bigcap_{i=1}^{n} V_i \in \tau$.
3. *If* $\{V_\alpha\}_\alpha$ *is a collection of elements of* τ *(finite, countable, uncountable), then* $\bigcup_\alpha V_\alpha \in \tau$.

Definition 8. *If τ is a topology in X, X is called a topological space. The members of τ are called the open sets in X.*

Definition 9. *If X, Y are topological spaces, then a function $f : X \to Y$ is continuous if for each open set $V \subset Y$, the set $f^{-1}(V)$ is also open.*

Definition 10. *Let Y be a subset of the topological space (X, τ_X). The induced topology defined by:*

$$\tau_Y := \{V \cap Y; V \in \tau\}$$

is a topology on Y.

For example, the topology of the unit sphere $\mathbb{R}^2$ is induced by the topology of $\mathbb{R}^3$.

Definition 11. *Let (X, τ_X) be a topological space, and $\sim$ an equivalence relation on X. The quotient set $Y := X_{/\sim}$ is the set of equivalence classes of elements of X. The quotient topology τ_Y on Y is defined by:*

$$\tau_Y := \{U \subseteq Y; \{x \in X; [x] \in U\} \in \tau_X\}.$$

For example, with $\mathbb{R}^2$, we define the topology on $\mathbb{P}^1(\mathbb{R})$.

Definition 12. *Let X be a space, and d a metric/distance on it. We define the open balls:*

$$B(x, r) := \{y \in X; \quad d(x, y) < r\}.$$

This set of balls defines a (base of) topology on X.

One example is $\mathbb{C}$, with the modulus as metric.

Day 2

1.0.3 The algebraic point of view: the complex projective line. [16]

The considered set is $\mathbb{C}^2_{\setminus\{0\}} := \{(z, \omega) \neq (0, 0)\}$. We define the following equivalence relation:

$$(z_1, \omega_1) \sim (z_2, \omega_2) \quad \text{if} \quad \exists \lambda \in \mathbb{C}^*, \quad (z_1, \omega_1) = \lambda(z_2, \omega_2).$$

The classes of equivalence are denoted by $[z; \omega]$.

Definition 13. *We denote the complex projective plane by $\mathbb{P}^1(\mathbb{C})$ (sometimes $\mathbb{P}(\mathbb{C})$) as ${\mathbb{C}^2_{\setminus\{0\}}}_{/\sim}$.*

Once again, we can notice that the set of classes of equivalence $\{[z; 1]; z \in \mathbb{C}\}$ is isomorphic to $\mathbb{C}$. The point (or line in $\mathbb{C}^2$) $[1, 0]$ is the point at infinity. We thus get:

$$\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C} \cup \{\infty\}.$$

This point of view is interesting when we will consider the linear transformations on the euclidean space, and how the induced maps act on the quotient space.

1.0.4 The Riemann sphere. [9, 1]

We define the unit sphere in $\mathbb{R}^3$:

$$S^2 := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3; \quad x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

We denote the north pole by $N = (0, 0, 1)$. We define the stereographic projection (from N):

$$\begin{array}{cccc} \pi : & S^2_{\backslash\{N\}} & \to & \mathbb{C}, \\ & (x_1, x_2, x_3) & \mapsto & \frac{x_1 + i x_2}{1 - x_3}. \end{array}$$

π is a homeomorphism. In fact, if $\pi(x_1, x_2, x_3) = x + iy$, by computing the modulus, we get: $|z|^2 = \frac{1+x_3}{1-x_3}$, thus $x_3 = \frac{-1+|z|^2}{1+|z|^2}$. By taking the real and imaginary parts, we get:

$$x_1 = \frac{2x}{1 + x^2 + y^2}, \quad x_2 = \frac{2y}{1 + x^2 + y^2}.$$

We proved the existence of an inverse, and that both functions are continuous, thus the stereogrqphic projection is an isomorphism.
Exercise : Considering π as a function from S^2 to the (real) plane $z = 0$, prove that for any point (x_1, x_2, x_3), the three points (x_1, x_2, x_3), N and $\pi(x_1, x_2, x_3)$ are on the same line.
We denote another point ∞, and extend the function π by $\pi(N) = \infty$.

Definition 14. *We denote the Riemann sphere (or extended plane), denoted by $\hat{\mathbb{C}}$ (or $\bar{\mathbb{C}}$, or $\mathbb{C}_\infty$) by $\mathbb{C} \cup \{\infty\}$, where the topology is such that π is still a homeomorphism.*

1.0.5 Topology

$\mathbb{C}$ is complete, with the topology induced by the metric $|\cdot|$.
We now need to precise the topology on $\hat{\mathbb{C}}$. We use the topology on S^2 induced by the one of $\mathbb{R}^3$. Then the open sets of $\hat{\mathbb{C}}$ are the sets $U \subset \hat{\mathbb{C}}$, such that:

- if $\infty \notin U$, then U is an open set of $\mathbb{C}$.
- If $\infty \in U$, then $U_{\backslash\{\infty\}}$ as a set of $\mathbb{C}$ is the complement of a compact set (draw a picture).

Proposition 15. *$\hat{\mathbb{C}}$ is compact: every infinite sequence in $\hat{\mathbb{C}}$ has a convergent subsequence.*

The proof is easy, since S^2 is compact, and π is a homeomorphism!
Notice finally that $\hat{\mathbb{C}}$ is also called the one-point compactification of $\mathbb{C}$.

Remark 16. *We can also define the topology on $\mathbb{P}^1(\mathbb{C})$ by the angle between two lines, and define the convergence of a sequence $[z_n, \omega_n] \to [z; \omega]$. Does this notion defines a distance? I let you think about it. We finally found that $\mathbb{P}^1(\mathbb{C})$ is isomorphic to $\hat{\mathbb{C}}$!*

2 Operations, differentiability and series.

2.1 Operations

The operations are well-defined on $\mathbb{C}$. What happens on $\hat{\mathbb{C}}$? We expect the following operations:

- $\forall z \in \mathbb{C}^*$, $z \cdot \infty = \infty$.
- $\forall z \in \mathbb{C}$, $z + \infty = \infty$.
- $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$.
- $\infty \cdot \infty = \infty$.

However, ∞ does not have an inverse : $\infty \cdot 0$ and $\infty + \infty$ are not defined.
To make the computations involving the point ∞ rigorous, we will consider any sequence converging to ∞; apply the formulae to each point of the sequence; check that this new sequence converges to a point in $\hat{\mathbb{C}}$. In other words, we use the topology of $\hat{\mathbb{C}}$ to properly define those operations.

2.2 Functions, differentiability.

In this section, U is considered as an open set (of $\mathbb{C}$ or $\hat{\mathbb{C}}$).

Definition 17. *A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is (complex-) differentiable at $z_0 \in U$ if the following quantity exists:*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and the limit is denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

Notice that the complex differentiability contains more information than the real differentiability on $\mathbb{R}^2$. The complex differentiability implies the real (on $\mathbb{R}^2$) one, but the converse is not true.
For example: if $f(z) = z$, then $f'(z) = 1$ (on $\mathbb{C}$). However, if $f(z) = \bar{z}$, this function is not differentiable. To prove it, consider the limit at the point $z_0 = 0$ along the real axis, the limit will be equal to 1, but the limit along the imaginary axis is -1.
At ∞, we use the function $j : z \mapsto \frac{1}{z}$, extended by $j(0) = \infty$ and $j(\infty) = 0$.

Definition 18. *Let U be an open set of $\hat{\mathbb{C}}$, with $\infty \in U$, and $0 \notin U$, and $f : U \to \mathbb{C}$. f is differentiable at ∞ if $f \circ j$ is differentiable at 0.*

The set of holomorphic functions on an open set $U \subset \mathbb{C}$ is denoted by:

$$H(U) = \{f : U \to \mathbb{C}; f \text{ differentiable at } z_0, \forall z_0 \in U\}.$$

For example, the function $f(z) = \frac{1}{z^2+1}$ is in $H(U)$, where $U = \mathbb{C}_{\{-i,i\}}$. Notice that we did not define yet functions with values in $\hat{\mathbb{C}}$!
Exercise: We have already defined the exponential. Find the range by the exponential map of the lines $x = 0$, $x = 1$, $x = 2$, $y = 0$, $y = 1$ and $y = \frac{\pi}{2}$.

Proposition 19. *If $f \in H(U)$, with $z = x + iy$, we define $u(x, y) := Re(f(x + iy))$, and $v(x, y) := Im(f(x + iy))$, and thus $f(x + iy) = u(x, y) + iv(x, y)$. Then f satisfies the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

In other words, the previous relation can be written by $\frac{\partial f}{\partial \bar{z}} = 0$. We will see this notation appear later.

2.3 Power series

In this part, U is an open subset of $\mathbb{C}$.

Definition 20. *A function f is analytic at $z_0 \in U$ if there exists $r > 0$, and coefficients $(a_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that the serie:*

$$\sum_{n=0}^{+\infty} a_n (z - z_0)^n$$

converges absolutely on the disk $D(z_0, r)$ centered at z_0 and of radius r. We have:

$$\forall z \in D(z_0, r), \quad f(z) := \sum_{n=0}^{+\infty} a_n (z - z_0)^n.$$

f is analytic on U if it is analytic on any point of U.

Proposition 21. *The radius of convergence of a power serie is:*

$$R := \left(\limsup_n |a_n|^{\frac{1}{n}} \right)^{-1}.$$

For example, the sequence $\sum_{n=1}^{+\infty} \frac{(1-z)^n}{n}$ has a radius $R = 1$ (prove it!) This radius is coherent with what we know on the real axis: if you consider this sequence on reals, we find the logarithm, which is not defined at 0.

Theorem 22. *If f is analytic on U, then $f \in H(U)$. Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ on $D(z_0, r)$, then :*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Proof. We define the function $g(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$. We can prove (try to write it) that $R_g = R_f$, and then g is well-defined on $D(z_0, R)$.
To prove the differentiability, let $\omega \in D(z_0, r)$, and $\rho > 0$ such that $|\omega| < \rho < r$ (we chose $z_0 = 0$ for simplicity). If $z \neq \omega$:

$$\begin{aligned}
\frac{f(z) - f(\omega)}{z - \omega} - g(\omega) &= \sum_{n=1}^{\infty} a_n \left(\frac{z^n - \omega^n}{z - \omega} - n\omega^{n-1} \right) \\
&= \sum_{n=2}^{\infty} a_n \left((z - \omega) \sum_{k=1}^{n-1} k\omega^{k-1} z^{n-k-1} \right) \\
\left| \frac{f(z) - f(\omega)}{z - \omega} - g(\omega) \right| &\leq |z - \omega| \sum_{n=2}^{\infty} a_n n^2 \rho^{n-1} \underset{\omega \to z}{\to} 0.
\end{aligned}$$

□

Corollary 23. *If f is analytic on U, by induction, f has a derivative at all order:*

$$\forall z \in D(z_0, r), \quad f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}.$$

Furthermore:

$$\forall n, \quad a_n = \frac{1}{n!} f^{(k)}(z_0).$$

3 Integration

To compute a complex integral, we need first to define the comdain of integration of the function. After defining the paths, we investigate how deep the integration is : the independence of the integral of a complex function along a path is deeply related to the property of being holomorphic. The local notion of integrability is related to the local notion of holomorphy (or analyticity).
If a function possesses a singularity and if a path goes around the singularity, the integral of the function along the closed path may not be equal to zero. This phenomenon will be investigated in this chapter, and we study the particular case of the index theory.
The local notions like the local Cauchy formula will then be generalized to a global level, by the homology theory.
Finally, we will get an overview of the different possibilities of singularities. This classification relies on the previous global notions of integration.

The notions of this part are local, thus we can focus on open sets of $\mathbb{C}$.

Appendix : Weekly recall of topology [11, 18]

Definition 24. *A curve $\gamma : [a,b] \to \mathbb{C}$ is a $\mathcal{C}^1$ function, where $[a,b]$ is a real interval. By $\mathcal{C}^1$, we mean that the real functions γ_{Re} and γ_{Im} such that $\gamma = \gamma_{Re} + i\gamma_{Im}$ are $\mathcal{C}^1$. $\gamma(a)$ is called the beginning point, and $\gamma(b)$ is called the end point.*
A path $\gamma : [a,b] \to \mathbb{C}$ is a continuous function, such that there exists a decomposition $a = a_0 \leq a_1 \leq \cdots \leq a_n = b$, for which the restriction of γ to each interval $\gamma_i := \gamma_{|[a_i;b_i]}$ is a curve. We thus authorized a finite number of points for which γ is not differentiable, and we have $\gamma_i(a_{i+1}) = \gamma_{i+1}(a_{i+1})$.
A curve or a path γ is said to be closed if $\gamma(a) = \gamma(b)$.
The range of a path γ, denoted by γ^* is a subset of $\mathbb{C}$: $\gamma^* := \gamma([a,b])$.

For example, consider the curve $\gamma(t) = e^{it}$ for $t \in [0,\pi]$. The end points are 1 and -1 in the complex plane.

In any picture, we denote by an arrow the sense in which we follow the range of a path. Notice that two paths can have the same range, with different properties (sense, velocity...). Later, when we will define the integral, we will impose some restrictions on the set of functions to integrate to avoid this dependency on the path.

Definition 25. *A subset U of $\mathbb{C}$ is pathwise connected (or path-connected) if for any two points α, β in U, there exists a path γ into U with $\gamma(a) = \alpha$ and $\gamma(b) = \beta$.*
A subset U of $\mathbb{C}$ is connected if there does not exist any couple (V, W) of open sets of U satisfying:

- *V and W are non empty.*
- *$V \cup W = U$.*
- *$V \cap W = \emptyset$.*

In particular, a pathwise connected set is also a connected set. To prove it, consider a pathwise connected set which is not connected. Let V, W be two open sets satsfying the three previous points. Let $\alpha \in V$, $\beta \in W$. There exists a path $\gamma : [a,b] \to U$ with end points α and β. By assumption, $\gamma^* \subset V \cup W$. Consider $z_0 := f(\gamma(\tilde{a}))$, where $\tilde{a}$ is the supremum of $t \in [a,b]$ of points t such that $\gamma(t) \in W$. By the open properties, $z_0 \notin V$, $z_0 \notin W$, which is absurd.
The other implication does not hold, for example consider the following comb. Let $I_n := \left\{\frac{1}{n} + iy; y \in [0,1]\right\}$ be subsets of $\mathbb{C}$ for $n \in \mathbb{Z}_{\backslash 0}$, and $I_0 = \{iy; y \in [0,1]\}$, and $J = \{x + 0i; x \in [-1;1]\}$. Now the comb $U := \left(\bigcup_{n\in\mathbb{Z}} I_n \cup J\right)_{\backslash 0}$ is connected, but not pathwise connected. However, we can consider a subset of the sets of $\mathbb{C}$ so that the two previous definitions are equivalent:

Proposition 26. *Let $U \subset \mathbb{C}$ be open. If U is connected, then it is also path-connected.*

Proof. Let $a \in U$, and consider the set $V \subset U$ composed of points which can be connected by a path in U. V is open, since U has the topology of $\mathbb{C}$: any point $b \in V$ is contained in a ball $B(b, r_b) \subset U$, and each point in this ball is connected to b. Thus if $b \in V$, so is a neighbourhood of b. Its complement is also open, for the same reason. We have $V \cup W = U$, and $V \cap W = \emptyset$, thus by the definition of being connected, $V = \emptyset$ or $W = \emptyset$. But $a \in V$, so any point can be connected to a. ☐

Definition 27. *A region U is an open and connected subset of $\mathbb{C}$.*

3.1 Homotopy [11]

In this part, we consider U an open subset of $\mathbb{C}$.

Definition 28. *Let γ, η be two paths in U, with $\gamma, \eta : [a,b] \to U$. γ is homotopic to η if there exists a continuous function $\psi : [a,b] \times [c,d] \to U$, with $[c,d]$ a real interval, and $\psi(t,c) = \gamma(t)$ and $\psi(t,d) = \eta(t)$.*

A picture of a homotopy.
In the case that $\gamma(a) = \eta(a)$ and $\gamma(b) = \eta(b)$, we will use that the homotopy ψ leaves the end points invariant if for any s, $\psi(a,s) = \gamma(a)$ and $\psi(b,s) = \gamma(b)$. We also consider that for each s, $\psi(\cdot, s)$ is a path.
A picture of non-homotopic paths.

Definition 29. *Let U be an open set. U is simply connected if it is connected and every closed path is homotopic to a point, or in other words, if two paths with same end points are homotopic.*

3.2 Integral over paths

Consider a continuous function $F : [a;b] \to \mathbb{C}$, with the decomposition with real and imaginary parts $F(t) = u(t) + iv(t)$. We define the integral of F by :

$$\int_a^b F(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

We can now define the integral along a path. The situation is a bit different from the real case : being continuous is not sufficient to define a primitive, we need an additional condition. In fact, the integral along a path highly depends on the path taken.

Definition 30 ([18])**.** *We define the integral of a function $f \in \mathcal{C}(U, \mathbb{C})$ along:*

- *a curve $\gamma : [a,b] \to U$ by*

$$\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

- *a path $\gamma : [a,b] \to U$ by*

$$\int_\gamma f = \sum_{i=1}^{n} \int_{a_i}^{a_{i+1}} f(\gamma_i(t))\gamma_i'(t)dt,$$

  where γ_i are curves. For short, we may also use the notation $\int_a^b f(\gamma(t))\gamma'(t)dt$ for paths.

Notice that this definition is coherent; if we take another decomposition $a = b_0 \leq b_1 \leq \cdots \leq b_m = b$ of the path, we obtain the same integral.

Remark 31. *If $f(x+iy) = u(x,y) + iv(x,y)$ with u and v real function, and $\gamma = \gamma_{Re} + i\gamma_{Im}$ is curve, we get:*

$$\int_a^b f(\gamma)\gamma' = \int_a^b u(\gamma_{Re}, \gamma_{Im})\gamma_{Re}' - v(\gamma_{Re}, \gamma_{Im})\gamma_{Im}' + i\int_a^b u(\gamma_{Re}, \gamma_{Im})\gamma_{Im}' + v(\gamma_{Re}, \gamma_{Im})\gamma_{Re}'.$$

Is there any link between this formula and the Cauchy-Riemann equation?

Remark 32. *Consider the curve $\gamma : [0, b-a] \to \mathbb{C}$, and the curve $\gamma^- : [b-a, 0] \to \mathbb{C}$ with $\gamma^-(t) = \gamma(b-t)$. Draw a picture, and we obtain by a change of variables:*

$$\int_{\gamma^-} f = -\int_\gamma f.$$

Example: Compute the integral along $\gamma(t) = re^{it}$, with $0 \leq t \leq 2\pi$, of $f(z) = z^2$. (Compute $f(x+iy)$).

Theorem 33 (Local Cauchy Theorem on a ball/ convex set, [11])**.** *Let U be a ball/ a convex set, and $f \in H(U)$. For any closed curve γ on U, we get $\int_\gamma f = 0$. Furthermore, f has a primitive on U : $F \in H(U)$, such that $F' = f$.*

How can we extend this theorem to a region?
If $\gamma : [a,b] \to U$ where U is a region, we can decompose $a = a_0 \le \cdots \le a_n = b$ such that for any $i \in [\![0, n-1]\!]$, there exists a radius $r_i > 0$, a point $\omega_i \in U$ such that $\gamma([a_i, a_{i+1})$ is a subset of the disk $D_i := D(\omega_i, r_i)$ (this property is true, we use that the image of a compact by a continuous function is compact). We define the integral of f along γ by:

$$\int_\gamma f := \sum_{i=0}^{n-1} \int_{\gamma_i} f.$$

The integral over γ_i depends on γ, but also on the chosen disks. Or does it? It is not clear that this integral is well-defined. By defining g_i a primitive of f on D_i, we get:

$$\int_\gamma f = \sum_{i=0}^{n} \left[g(\gamma_i(a_{i+1}) - g(\gamma(a_i))\right].$$

Theorem 34 ([11])**.** *The previous defined integral is independent of the chosen disks. Moreover, if γ and η are homotopic with the same end points, then:*

$$\int_\gamma f = \int_\eta f.$$

For example: consider the punctured complex plane $\mathbb{C}_{\setminus 0}$. On $[0,\pi]$, consider the paths $\gamma : t \mapsto e^{it}$, $\tilde{\gamma} : t \mapsto \cos(t) + 2i\sin(t)$, $\eta : t \mapsto e^{-it}$. Now consider the functions $f(z) = \frac{1}{z}$, and $\tilde{f}(z) = \frac{1}{z^2}$. Compute the different possibilities of integrals of functions along the different paths. What can you notice?

Proof. First, we prove that for fixed function and path, the integral is independent of the disks nor the decomposition. Consider the decomposition $a = a_0 \le \cdots \le a_n = b$ (respectively $a = \tilde{a}_0 \le \cdots \le \tilde{a}_m = b$), the set of disk $D(\omega_i, r_i)$ (resp. $D(\tilde{\omega}_j, \tilde{r}_j)$) and a primitive on each disk g_i (resp. $\tilde{g}_j$).
We can find another decomposition of the segment $a = c_0 \le \cdots \le c_p = b$, where each a_i and $\tilde{a}_i$ is contained in $(c_k)_{k \le p}$. In particular, for each $k \le p$, there exist unique indices i_k and j_k such that $[c_k; c_{k+1}] \subset [a_{i_k}, a_{i_k+1}]$ and $[c_k; c_{k+1}] \subset [\tilde{a}_{j_k}, \tilde{a}_{j_k+1}]$. We get by the Cauchy theorem:

$$\begin{aligned}\int_{\gamma_{|[c_k;c_{k+1}]}} f &= g_{i_k}(\gamma(c_{k+1})) - g_{i_k}(\gamma(c_k)) \\ &= \tilde{g}_{j_k}(\gamma(c_{k+1})) - \tilde{g}_{j_k}(\gamma(c_k)).\end{aligned}$$

By summing to obtain the integral, we get:

$$\begin{aligned}\sum_{k\ge 0} g_{i_k}(\gamma(c_{k+1})) - g_{i_k}(\gamma(c_k)) &= \sum_{k \ge 0} \tilde{g}_{j_k}(\gamma(c_{k+1})) - \tilde{g}_{j_k}(\gamma(c_k)) \\ \sum_{i \ge 0} g_i(\gamma(a_i)) - g_i(\gamma(a_i)) &= \sum_{j\ge 0} \tilde{g}_j(\gamma(\tilde{a}_{j+1})) - \tilde{g}_j(\gamma(\tilde{a}_j)),\end{aligned}$$

where the last line comes from a simplification : if the primitive g_j is the same on the intervals $[c_k, c_{k+1}]$ and $[c_{k+1}; c_{k+2}]$, then the terms at c_{k+1} compensate.
Now we prove that two homotopic paths γ and η will define the same integral. Consider the homotopy ψ, with $\psi(a,s) = \gamma(a)$, $\psi(b,s) = \gamma(b)$, $\psi(t,c) = \gamma(t)$ and $\psi(t,d) = \eta(t)$.
We claim that we can cut $[a,b] \times [c,d]$ into rectangles $[a_i; a_{i+1}] \times [c_j; c_{j+1}]$ where $a = a_0 \le \cdots \le a_n = b$ and $c = c_0 \le \cdots \le c_m = d$ such that $\psi([a_i, a_{i+1}] \times [c_j, c_{j+1}])$ is a subset of a disk of radius r in U.
It is sufficient to prove that:

$$\forall j, \quad \int_{\psi(\cdot, c_j)} f = \int_{\psi_{\cdot, c_{j+1}}} f.$$

In fact, the proof of this equality is similar to the one done before, since:

$$g_{i,j+1}(\psi(a_{i+1}, c_{j+1})) - g_{i,j+1}(\psi(a_i, c_{j+1})) = g_{i,j}(\psi(a_{i+1}, c_j)) - g_{i,j}(\psi(a_i, c_j))$$

(Make a picture!)
Ideas of proof of the claim:
Denote by R the rectangle $[a,b] \times [c,d]$. Since ψ is continuous, there exists $\delta > 0$ such that the distance between $\psi(R)$ (compact) and the boundary of U is larger than δ (Weierstrass).

Let $\epsilon > 0$ (useful to consider open sets on the compact set R). Consider for any n, the set of open rectangles :

$$\forall 0 \leq j,k \leq 2^n - 1,$$
$$R_{n,j,k} := R \cap \left(a + \frac{b-a}{2^n}j - \frac{\epsilon}{2^n}; a + \frac{b-a}{2^n}(j+1) + \frac{\epsilon}{2^n}\right) \times \left(c + \frac{d-c}{2^n}j - \frac{\epsilon}{2^n}; c + \frac{d-c}{2^n}(k+1) + \frac{\epsilon}{2^n}\right).$$

For a fixed (n,j,k), we denote $\delta_{n,j,k}$ the diameter of $\psi(R_{n,j,k})$, and a point $\omega_{n,j,k} \in \Psi(R_{n,j,k})$. Consider thus the family:

$$\mathcal{F} := \{D_{n,j,k} := D(\omega_{n,j,k}, \delta_{n,j,k}); 2\delta_{n,j,k} < \delta\}.$$

We claim that for any $(t,s) \in R$, there exists a rectangle $R_{n,j,k} \ni (t,s)$, and for those indices, $D_{n,j,k} \in \mathcal{F}$. Otherwise, let $(t_0, s_0) \in R$, such that for any $R_{n,j,k} \ni (t_0, s_0)$ we have $D_{n,j,k} \notin \mathcal{F}$. The set $B(\psi(t_0,s_0), \frac{\delta}{2}) \cap \psi(R)$ is open in $\psi(R)$, and its preimage is open. In particular, it contains an open ball $B((t_0,s_0), r)$ in R, and thus a rectangle $R_{n,j,k}$ for n large enough. This rectangle satisfies $D_{n,j,k} \in \mathcal{F}$, which is absurd. Then the family of open rectangles $R_{n,j,k}$ in R covers the compact R, and we can then extract a finite covering. By taking n_0 the smallest n, the rectangles $\{R_{n_0,j,k}\}_{j,k}$ cover R, and the image of each of this rectangle is in an expected ball. Now, take the same rectangles without ϵ, we obtain the expected covering. ☐

Example [11] : Principal value of logarithm. We define the following function:

$$\begin{array}{rccc} Log: & U & \to & \mathbb{C} \\ & z & \mapsto & \int_1^z \frac{1}{\omega} d\omega, \end{array}$$

where $U = \left\{re^{i\theta}; -\pi < \theta < \pi, r > 0\right\}$. If $z = Re^{i\theta}$:

$$\begin{aligned} Log(z) &= \int_1^R \frac{1}{\omega} d\omega + \int_R^{Re^{i\theta}} \frac{1}{\omega} d\omega \\ &= \ln(R) + \int_{\phi=0}^{\theta} \frac{ire^{i\phi}}{re^{i\phi}} d\phi = \ln(R) + i\theta. \end{aligned}$$

This function satisfies $e^{Log(z)} = z$.
Finally, notice that we can also define this function on a other domain, for example with the angles $\theta \in (0, 2\pi)$.

Day 4

Remark 35. *If U simply connected, then any $f \in H(U)$ admits a primitive.*
For a disk $D(z_0, \rho)$, we denote by the contour the curve: $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = \rho e^{it} + z_0$.

Theorem 36 (Local Cauchy formula[17])**.** *Let U be an open set, $\rho > 0$, and $\overline{D(z_0, \rho)} \subset U$. Let γ be the contour; then:*

$$\forall z \in D(z_0, \rho), \quad f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\omega)}{\omega - z} d\omega.$$

Proof. By a keyhole. The function $\omega \mapsto \frac{f(\omega)}{\omega - z}$ is holomorphic in the all set, except at z. Define a closed path, with four curves : a first γ_0 is almost the contour of $D(z_0, R)$, with $R < \rho$; a second γ_2 is almost the contour of $D(z, r)$, with r small; the two other paths, γ_1 and γ_3 making the link between the two others, whose range is at a distance ϵ from another range. Now let ϵ goes to 0, we obtain $\int_{\gamma_1} g = -\int_{\gamma_3}$ (use the Cauchy theorem and that $\frac{f(\omega)}{\omega - z}$ is bounded on the considered set, thus with $r > 0$. By the Cauchy theorem, we get:

$$0 = \int_{\gamma_0} \frac{f}{\omega - z} + \int_{\gamma_1} \frac{f}{\omega - z} + \int_{\gamma_2} \frac{f}{\omega - z} + \int_{\gamma_3} \frac{f}{\omega - z} = \int_{\gamma_0} \frac{f}{\omega - z} + \int_{\gamma_2} \frac{f}{\omega - z}.$$

When R goes to ρ, $\int_{\gamma_0} \to \int_\gamma$. When r goes to 0, we get:

$$\int_{\gamma_2} \frac{f(\omega)}{\omega - z} d\omega = \int_{\gamma_2} \frac{f(\omega) - f(z)}{\omega - z} d\omega + f(z) \int_{\gamma_2} \frac{1}{\omega - z} d\omega = o(r) + 2\pi f(z).$$

Notice that another proof [11], uses that γ_0 and γ_2 are homotopic, so the integrals are equal. ☐

Theorem 37 ([18, 11])**.** *Let f be holomorphic on the closed disk $\overline{D(z_0,R)}$ with $R>0$, and γ the contour. Then f has a power serie expansion:*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n, \qquad \text{with} \qquad a_n=\frac{1}{n!}f^{(n)}(z_0)=\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{(\omega-z_0)^{n+1}}d\omega$$

Furthermore:

$$|a_n|\leq \frac{1}{R^n}\sup_{|z-z_0|=R}|f(z)|,$$

and the radius of convergence of the serie is larger than R.

Proof. Let $\omega\in\gamma^*$, where γ is the contour. We have:

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega-z}d\omega.$$

Let $0<r<R$. What is the power serie expansion on $D(z_0,r)$?

$$\forall z\in D(z_0,r), \quad \frac{1}{\omega-z}=\frac{1}{\omega-z_0}\frac{1}{1-\frac{z-z_0}{\omega-z_0}}=\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\omega-z_0)^{n+1}},$$

where the last line is valid since we have the uniform convergence ot the geometric serie:

$$\left|\frac{z-z_0}{\omega-z_0}\right|\leq\frac{r}{R}<1.$$

Thus, by uniform convergence again, we get:

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega-z}d\omega=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{(\omega-z_0)^{n+1}}(z-z_0)^n d\omega.$$

□

Corollary 38. *An holomorphic function is analytic.*

Definition 39. *An entire function is a holomorphic function on $\mathbb{C}$. We thus get the convergence of any power serie on $\mathbb{C}$.*

Proposition 40 ([11])**.** *Let f be an entire function, and suppose that there exists $k\in\mathbb{N}$, $C>0$, such that:*

$$\forall R>0, \quad \sup_{|z-z_0|=R}|f(z)|\leq CR^k.$$

Then f is a polynomial of degree $\leq k$.

Corollary 41 (Liouville's theorem)**.** *A bounded entire function is constant.*

Theorem 42 (Morera's theorem)**.** *Let U be an open set in $\mathbb{C}$, and let $f\in\mathcal{C}(U,\mathbb{C})$. Suppose that for any closed curve γ on U, we have $\int_\gamma f=0$. Then f is analytic.*

Proof. If $\int_\gamma f=0$, we can, as for the Cauchy's theorem, define a primitive g. This function is differentiable, and holomorphic, with derivative f. In particular, g is analytic, and so is its derivative f. □

3.3 Zeros of analytic functions

Let U be an open set, and $f\in H(U)$. We define the set of zeros of f :

$$\mathcal{Z}(f):=\{z\in U; f(z)=0\}.$$

Proposition 43 ([18])**.** *If $f\neq 0$, then $\mathcal{Z}(f)$ is discrete (wihtout limit point/ accumulation point <u>in</u> U.*

For example, $e^{iz}=e^{-iz}$ if and only if $Re(z)=0[\pi]$ and $Im(z)=0$. Thus:

$$\text{in } \mathbb{C}, \quad \mathcal{Z}(\sin)=\pi\mathbb{Z},$$
$$\text{in } \mathbb{C}^*, \quad \mathcal{Z}(\sin(\frac{1}{z}))=\left\{\frac{1}{\pi n}, n\in\mathbb{Z}^*\right\}.$$

In those examples, there is no accumulation point in the domains.

Proof. Suppose that $z_n \to z_0 \in U$, with $z_n \neq z_m$ for $n \neq m$, and $z_0 \in \mathcal{Z}(f)$. Since U is open, for r small enough, $D(z_0, r) \subset U$. Suppose that $f \neq 0$. Then, let m the smallest index such that $a_m \neq 0$ in the following serie:

$$f(z) = \sum_{n \leq m} a_n (z - z_0)^n = a_m (z - z_0)^m \left(\sum_{n \geq 0} \frac{a_{n+m}}{a_n} (z - z_0)^n \right) = a_m (z - z_0)^m (1 + h(z)).$$

In particular, by studying the radius of convergence, we obtain that $h \in H(D(z_0, r))$, with $h(z_0) = 0$, and for $\tilde{r} < r$ small enough:

$$\forall z \in D(z_0, \tilde{r}), \quad 1 + h(z_0) \neq 0.$$

However, for large n, $z_n \in D(z_0, \tilde{r})$ and $f(z_n) = 0$: it is absurd. □

Notice that if two holomorphic functions f and g are equal on a set with a limit pint, we have the equality of the two functions! This is the case on $(0, 2)$ for:

$$f(z) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} (z - 1)^n, \quad \text{and} \quad g(z) = Log(z).$$

Notice also that if $f \neq 0$, then $\mathcal{Z}(f)$ is finite on any compact set.
In the later (see for example [1]), we will consider compact subsets of U. If f can be factorized according to its zeros $z_1, \cdots z_n$, we get $f(z) = \prod_i (z - z_i) g(z)$, with as expected g holomorphic and without zeros on this compact. By computing the logarithmic derivative, we get:

$$\frac{f'}{f} = \sum_{i=1}^{n} \frac{1}{z - z_i} + \frac{g'}{g}.$$

We need to be able to compute the integral of $\frac{1}{\omega}$ for any path.

Definition 44 ([18])**.** *Let γ be a closed path on $\mathbb{C}$, and Ω the complement of γ^* in $\mathbb{C}$. We define the index (or winding number):*

$$\forall z \in \Omega, \quad Ind_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{d\omega}{\omega - z}.$$

Ind_γ is well-defined on Ω, and is an integer-valued function. It is constant on each connected component of Ω, and equal to 0 on the non-compact component.

Proof. Let $\gamma : [a, b] \to \mathbb{C}$ be a path, and for $z \in \Omega$, we define the function on $[a, b]$:

$$\phi : t \mapsto \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z} ds \right).$$

In fact, the previous integral is well-defined for a curve; for a general path, use the definition of the integral over a path. This function is differentiable, except at a finite set of points $S \subset [a, b]$, where γ is not differentiable, and we get:

$$\phi'(t) = \phi(t) \frac{\gamma'(t)}{\gamma(t) - z}.$$

Thus $\left(\frac{\phi}{\gamma - z}\right)' = 0$, and ϕ is constant on each interval of $[a, b]_{\setminus S}$. Because $\frac{\phi}{\gamma - z} \in \mathcal{C}([a, b], \mathbb{C})$, this function is constant. Because $\phi(a) = 1$ and ϕ is a closed path, we get:

$$\frac{\phi(t)}{\gamma(t) - z} = \frac{1}{\gamma(a) - z}, \quad \phi(b) = \frac{\gamma(b) - z}{\gamma(a) - z} = 1.$$

Thus there exists $n \in \mathbb{Z}$, such that:

$$2i\pi n = \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} ds, \quad \text{so } Ind_\gamma(z) \in \mathbb{Z}.$$

Day 5

Lemma 45 ([18]). *Let γ be a closed path, f continuous on U, and $\gamma^* \cap U = \emptyset$. Then the function:*

$$z \mapsto \int_\gamma \frac{d\omega}{f(\omega) - z}$$

is analytic on U.

Proof. Let $D(a, r) \subset U$, we thus get:

$$\forall z \in D(a, r), \quad \left| \frac{z - a}{f(\omega) - a} \right| \leq \frac{|z - a|}{r} < 1.$$

Thus:

$$\sum_{n=0}^{+\infty} \frac{(z-a)^n}{(f(\omega) - a)^{n+1}} = \frac{1}{f(\omega) - a} \frac{1}{1 - \frac{z-a}{f(\omega)-a}} = \frac{1}{f(\omega) - z},$$

and the serie converges uniformly to the sum on the previous disk. This convergence enables to interchange the integral and the sum:

$$\int_\gamma \frac{d\omega}{f(\omega) - z} = \sum_{n \geq 0} \int_\gamma \frac{(z-a)^n}{(f(\omega) - a)^{n+1}} d\omega = \sum_{n \geq 0} a_n (z - a)^n.$$

□

Then Ind_γ is continuous on each connected component of Ω, and thus constant. In particular, by:

$$2i\pi n = \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} ds \underset{|z| \to \infty}{\to} 0,$$

we get $n = 0$ for the "exterior" component. □

For example, consider the curve $\gamma : t \mapsto e^{it}$ defined on $[0; 2\pi]$. What is $Ind_\gamma(z)$ for any z such that $|z| \neq 1$? If $|z| > 1$, then $Ind_\gamma(z) = 0$. If $|z| < 1$, then $Ind_\gamma(z) = Ind_\gamma(0)$, so we compute:

$$Ind_\gamma(0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ie^{it}}{e^it} dt = 1.$$

It is thsu coherent with the notion of winding number.

Theorem 46 ([4]). *Let U be a simply-connected region, $f \in H(U)$ with zeros $\{z_0, \cdots, z_k\}$ and multiplicities $\{m_0, \cdots, m_k\}$. If γ is a curve in U, not passing through $\{z_0, \cdots, z_k\}$, then we can integrate the "logarithmic derivative:*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{i=1}^k m_i Ind_\gamma(z_i).$$

Draw an example, with for example, z_0 and z_1 multiplicity 1, and z_2 with multiplicity 2, and a curve γ going couter-clockwise around z_2, and clockwise around z_0. Then :

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = 1 \cdot (-1) + 1 \cdot 0 + 2 \cdot 1 = 1.$$

Proof. We define the following function:

$$g : z \mapsto \begin{cases} \dfrac{f(z)}{\prod_{i=1}^k (z - z_i)^{m_i}} & \text{if } z \neq z_i \\ \dfrac{1}{\prod_{j \neq i} (z - z_j)^{m_j}} \dfrac{f^{(m_i)}(z_i)}{m_i!} & \text{if } z = z_i. \end{cases}$$

The function g is continuous, but is also holomorphic. In fact, it is holomorphic on $U_{\setminus \{z_1, \cdots z_k\}}$ (easy), and for each z_i, f can be developped into a power serie at z_0, and this development gives one of g except at z_0:

$$f(z) = \sum_{n=m_i}^\infty a_n (z - z_i)^n = (z - z_i)^{m_i} \sum_{n=0}^\infty a_{n+m_i} (z - z_i)^n, \quad g(z) = \frac{1}{\prod_{j \neq i} (z - z_j)^{m_j}} \sum_{n=0}^\infty a_{m_i + n} (z - z_i)^n.$$

Thus g admits a power serie expansion on a neighbourhood of z_i, thus g is analytic on U. Furthermore, in a small neighbourhood of z_i, we have $g(z) = a_{m_i}(1 + h(z))$ with $|h(z)| \leq \frac{1}{2}$; thus $g \neq 0$.
We thus obtain the following factorization of f:

$$f(z) = \prod_{i=1}^{k} (z - z_i)^{m_i} g(z),$$

$$f'(z) = \sum_{i=1}^{k} m_i (z - z_i)^{m_i - 1} \prod_{j \neq i} (z - z_j)^{m_j} g(z) + \prod_{i=1}^{k} (z - z_i)^{m_i} g'(z).$$

The computation of the logarithmic derivative gives:

$$\forall z \in U_{\setminus \{z_1, \cdots, z_k\}}, \quad \frac{f'(z)}{f(z)} = \sum_{i=1}^{k} \frac{m_i}{z - z_i} + \frac{g'(z)}{g(z)}.$$

Because g is holomorphic and never equal to 0, the integral along a closed curve of $\frac{g'}{g} \in H(U)$ is equal to 0, which gives the theorem. $\square$

Corollary 47. *Let D be a disk, $f \in H(D)$, where the set $\{f(z) = \alpha\}$ for $\alpha \in \mathbb{C}$ is finite: $\{f(z) = \alpha\} = \{z_1, \cdots, z_k\}$, with multiplicities $\{m_1, \cdots, m_k\}$. If γ is a closed curve on D, with $\gamma^* \cap \{f = \alpha\} = \emptyset$, then:*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - \alpha} dz = \sum_{i=1}^{k} m_i Ind_\gamma(z_k).$$

The proof is straightforward with $g = f - a$ and the previous theorem.
Consider now the following example. If $\gamma : [a, b] \to D$ is a path in a disk D, and $f \in H(D)$, and the path $\eta := f \circ \gamma$. Let us denote by $\{z_1, \cdot, z_k\}$ the zeros of f. Consider now the function:

$$g : t \mapsto \int_a^t \frac{\eta'(s)}{\eta(s) - \alpha} ds - \int_a^t \frac{f'(\gamma(s))}{f(\gamma(s)) - \alpha} \gamma'(s) ds$$

(with the usual notation for the path : the integral on the right is a shortcut for the sum of $\int_{\gamma_i}$). The function $g \in \mathcal{C}([a, b], U)$ is differentiable everywhere except at a finite number point $S \subset [a, b]$, and $g'(t) = 0$ except on S. By continuity, g is constant, and $g = g(a) = 0$. Thus:

$$2\pi i Ind_\eta(\alpha) = \int_a^b \frac{\eta'(t)}{\eta(t) - \alpha} dt = \int_\gamma \frac{f'(z)}{f(z) - \alpha} dz = 2\pi i \sum_{i=1}^{n} m_i Ind_\gamma(z_i).$$

Theorem 48 ([4])**.** *Suppose that $f \in H(D(z_0, R))$, and $\alpha = f(z_0)$. Suppose $f - \alpha$ has a zero of order m at z_0. Then there exist $\epsilon > 0$, $\delta > 0$, such that for any $\omega \in D(\alpha, \delta)$, the equation $f(z) = \omega$ has exactly m simple roots in $D(z_0, \epsilon)$.*

As an example, consider the function $f(z) = (z - 1)^3$. 1 is a zero of order 3. If you consider now a point in the image of f, like $\omega = re^{i0}$ for r small enough, then there are three points close to 1 satisfying $f(z) = \omega$. They are simple roots.

Proof. Let $\epsilon > 0$, $\epsilon < \frac{R}{2}$. Since the zeros of $f - \alpha$ are isolated on the compact $\overline{D(z_0, \frac{R}{2})}$, there is only a finite number of them in the disk. We can thus choose ϵ small enough such that $f - \alpha$ has no other roots than z_0 in $D(z_0, 2\epsilon)$. By the same arguments applied on the derivative, ϵ can be chosen so that $f'(z) \neq 0$ on $0 < |z - z_0| < 2\epsilon$ (except at $z = z_0$ in case of multiplicity $m \geq 2$.
Let $\gamma(t) = z_0 + \epsilon e^{i\theta}$, with $0 \leq \theta \leq 2\pi$, and the curve $\eta = f \circ \gamma$. Since $\alpha \notin \eta^*$, there exists $\delta > 0$ such that $D(\alpha, \delta) \cap \eta^* = \emptyset$. Thus $D(\alpha, \delta \subset \mathbb{C}_{\setminus \eta^*}$, and by the previous discussion:

$$Ind_\eta(\alpha) = \sum_{i=1}^{k} m_i Ind_\gamma(z_i), \quad \text{where } (z_i)_{i \leq k} = \mathcal{Z}(f - \alpha).$$

In fact, each $Ind_\gamma(z_i)$ is either equal to 0 or 1 by definition of γ. Because of the definition of ϵ, the only index which is different from 0 is the one of z_0.
Now, for any $\omega \in D(\alpha, \delta)$ (thus in the same connected component):

$$Ind_\eta(\alpha) = m = Ind_\eta(\omega).$$

The sum of the multiplicities of the roots of $\{f(z) = \omega\}$ in $D(z_0, 2\epsilon)$ is thus equal to m. However, all the roots are simple, because $f \neq 0$ on $D(z_0, 2\epsilon)_{\setminus \{z_0\}}$, thus $f(z) - \omega = 0$ has exactly m simple roots. $\square$

Corollary 49 (Open Mapping Theorem,[4]). *Let U be a region, $f \in H(U)$, and f non constant. The image of any open set by f is open.*

Proof. Let O be an open susbet of U. Let $\alpha \in O$. We can define $\epsilon > 0$ and $\delta > 0$ satisfying the assumptions of the previous theorem. Each point of $D(f(\alpha), \delta)$ has (at least) one preimage in $D(\alpha, \epsilon)$. $\square$

A function f satisfying this property is called an open-mapping.

Corollary 50 (Inverse mapping theorem,[4]). *Let U be a region, $f \in H(U)$, one-to-one (injective). Then the inverse $f^{-1} : f(U) \to U$ is analytic, and $(f^{-1})'(\omega) = \frac{1}{f'(z)}$ where $\omega = f(z)$.*

Proof. By the OMT, f^{-1} is continuous, $f(U)$ is open in $\mathbb{C}$. Since $z = f^{-1}(f(z))$, the derivative gives us the result. $\square$

Theorem 51 (Maximum modulus theorem,[1]). *If $f(z)$ is analytic and non constant on a region U, then $|f(z)|$ has no maximum in U.*

Proof. Let $z_0 \in U$. Then for some ϵ and δ, $D(z_0, \epsilon) \subset U$ and $D(f(z_0), \delta) \subset f(D(z_0, \epsilon))$. Thus $|f(z_0)|$ can not be a maximum. $\square$

Corollary 52. *Let U be a bounded region, $f \in H(U)$, and f continuous on $\bar{U}$ (the closure). The maximum of $|f|$ is achieved on the boundary of U.*

Notice that the proofs of the previous theorem can be done in different ways. For example, [11] does not involve the property of complex integration : those properties can be deduced from the theory of series. With a totally different point of view, [18] uses the decomposition of a function into its Fourier serie to obtain the maximum modulus theorem.

Day 6

Weekly recall of topology [18]

Definition 53. *Let U be an open set. The set of continuous functions $\mathcal{C}(U, \mathbb{C})$ can be equipped with the following topology: a sequence of functions $(f_n)_n$ converges uniformly on compact subsets to f if for any compact $K \subset U$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $n > N$, $\sup_{z \in K} |f_n(z) - f(z)| < \epsilon$.*

For example, consider the sequence of functions $f_n(z) = z^n$ on the unit disk $D(0, 1)$. This sequence of functions converges uniformly on any compact set to 0, but this convergence is not uniform on the all disk. In fact, the set $\mathcal{C}(U, \mathbb{C})$ of continuous functions on an open set U can be equipped with a metric, see [4].

Theorem 54. *Let U be open. If for any n, $f_n \in H(U)$ and f_n converges to a function f in the previous sense, then $f \in H(U)$. Furthermore, the sequence $(f_n')_n$ converges to f' in the previous sense.*

Proof. For a compact K, the sequence of continuous functions $(f_n)_n$ converges uniformly on this compact to f, so f is continuous. Thus $f \in \mathcal{C}(U, \mathbb{C})$.
Let us now consider a point $z_0 \in U$, and a disk $D := D(z_0, r) \subset U$. Let γ be a closed curve on D, and $\int_\gamma f_n = 0$. By uniform convergence on the compact set γ^*, we get $\int_\gamma f = 0$. Thus by Morera theorem, we get that $f \in H(D)$, and so $f \in H(U)$. (The previous proof does not work on the all set U since it may be not simply connected.)
Now we prove the uniform convergence of the derivatives. Let $z_0 \in U$, and $D(z_0, 2R) \subset U$. Consider the compact set $\overline{D(z_0, R)}$, and $z \in D(z_0, \frac{R}{2})$. Let $\mathcal{C}_{z,R}$ the circle centered at R and of radius R. We then have, by the Cauchy formula:

$$(f_n - f)'(z) = \frac{1}{2\pi i} \int_{C_{z,R}} \frac{(f_n - f)(\omega)}{(\omega - z)^2} d\omega.$$

In particular, $|\omega - z| > \frac{R}{2}$, and f_n converges uniformly on $\overline{D(z_0, \frac{3}{2}R)}$. Thus f_n' converges uniformly to f'. (Take care that we have not proved yet the Cauchy formula with disks which are not centered at z_0; this is why we need those pirouettes with disks.)

$\square$

Consider the following serie:

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \text{where } n^z = \exp\left(z \ln(n)\right).$$

Each term of the sum is well-defined, and the partial sums converges uniformly on the area $Re(z) > 1$. Furthermore, the derivative is:

$$f'(z) = \sum_{n=1}^{\infty} \frac{-\ln(n)}{n^z}.$$

3.4 Global Cauchy theory [11, 18]

Consider U an open set. The following definitions are dependent on the chosen set.
A chain : $\Gamma := \gamma_1 \dotplus \gamma_2 \dotplus \cdots \dotplus \gamma_n$ is a (formal) sum of paths γ_i in U.
A cycle is a chain where each γ_i is closed.
We can define the integral along a chain $\Gamma = \gamma_1 \dotplus \gamma_2 \dotplus \cdots \dotplus \gamma_n$:

$$\int_\Gamma f := \sum_{i=1}^{n} \int_{\gamma_i} f.$$

The definition of the index can also be extended to cycles. If $\Gamma = \gamma_1 \dotplus \gamma_2 \dotplus \cdots \dotplus \gamma_n$ is a cycle and z is not in the range of any γ_i, then:

$$Ind_\Gamma(z) := \int_\Gamma \frac{1}{\omega - z} d\omega.$$

We say that two closed curves (closed paths/ cycles) Γ and $\tilde{\Gamma}$ are homologous if for any $\alpha \notin U$, $Ind_\Gamma(\alpha) = Ind_{\tilde{\Gamma}}(\alpha)$. In this case we denote $\Gamma \sim \tilde{\Gamma}$. In particular, Γ is homologous to 0 in U, denoted $\Gamma \sim 0$ if $Ind_\Gamma(\alpha) = 0$ for any $\alpha \notin U$.
It is important to see the difference between being the notions of homology and the one of homotopy for paths. In fact, we get:

Proposition 55. *If γ is homotopic to a point, then it is homologous to* 0.

This proof is straightforward, just have on the index.
However, the converse is not true. By using some examples with open sets U with one or two holes, check that some closed paths homologous to 0 can be homotopic to a point, but can also be not homotopic to a point.

Theorem 56 (Global Cauchy theorem)**.** *Let $f \in H(U)$, with U an open set of $\mathbb{C}$. If Γ is a cycle in U, such that Γ is homologous to* 0 *then:*

$$f(z) \cdot Ind_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - z} d\omega, \qquad \text{for } z \in U_{\setminus \Gamma^*},$$

and

$$\int_\Gamma f(z)dz = 0.$$

If Γ_0 and Γ_1 are homologous, then:

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

We admit the proof of this theorem, since the ideas of the proof have been already seen in other proofs.
For example, consider an exotic path Γ in an open set U, where U has three holes which are points z_0, z_1 and z_2. Consider the paths $\gamma_i(t) = z_i + re^{i\theta}$ on $[0, 2\pi]$, for r small enough so that there is no intersection. Then the path Γ can be decomposed as:

$$\Gamma \sim Ind_\Gamma(z_0)\gamma_0 \dotplus Ind_\Gamma(z_1)\gamma_1 \dotplus Ind_\Gamma(z_2)\gamma_2.$$

For more on the theory concerning homotopy, see the reference [6]

3.5 Singularities [11]

Let $(a_n)_n \in \mathbb{C}^\mathbb{Z}$ be complex numbers. The Laurent serie is the following serie:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

Let U be a set.We say that the Laurent serie converge absolutely (resp. uniformly) if the two series $\sum_{n\geq 0} a_n z^n$ and $\sum_{n<0} a_n z^n$ converge absolutely (resp. uniformly) on U. f is the sum of the series. For $0 \leq r < R$, we define the annulus:

$$\mathcal{A}_{r,R} := \{r < |z| < R\}.$$

If $r = 0$, $\mathcal{A}_{r,R}$ is called a punctured disk. We also denote by $\mathcal{C}_r$ the circle centered at 0 and of radius r (counter-clockwise).

Theorem 57. *Let $0 < r < R$, and $f \in H(\mathcal{A}_{r,R})$. Then f has a Laurent expansion:*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n,$$

with

$$a_n := \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{f(\omega)}{\omega^{n+1}} dw \quad \text{for } n \geq 0,$$

and

$$a_n := \frac{1}{2\pi i}\int_{\mathcal{C}_r} \frac{f(\omega)}{\omega^{n+1}} dw \quad \text{for } n < 0.$$

This serie converges uniformly on $\mathcal{A}_{s,S}$ for $r < s < S < R$.

Day 7

Proof. Let $\epsilon > 0$ small enough, and consider the open set $U := \{r - \epsilon < |z| < R + \epsilon\}$. Then $\mathcal{C}_R - \mathcal{C}_r$ is homologous to 0 : if $|\alpha| < r - \epsilon$, $Ind_{\mathcal{C}_r}(\alpha) = Ind_{\mathcal{C}_R}(\alpha) = 1$, and if $|\alpha| > R + \epsilon$, $Ind_{\mathcal{C}_r}(\alpha) = Ind_{\mathcal{C}_R}(\alpha) = 0$. By the global Cauchy theorem:

$$\forall z \in \mathcal{A}_{r,R}, \quad f(z) \cdot Ind_{\mathcal{C}_R - \mathcal{C}_r}(z) = \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{f(\omega)}{\omega - z} d\omega - \frac{1}{2\pi i}\int_{\mathcal{C}_r} \frac{f(\omega)}{\omega - z)} d\omega.$$

It suffices now to develop each integral into series. The first integral has already been dealt with, in the case of power serie expansion.

- For $\mathcal{C}_R$, $|\omega| = R$, and $|z| = S$ with $S < R$:

  $$\frac{1}{\omega - z}\frac{1}{\omega}\frac{1}{1 - \frac{z}{\omega}} = \frac{1}{\omega}\sum_{n=0}^{\infty} \frac{z^n}{\omega^n}.$$

  Furthermore, the convergence is uniform since $|\frac{z}{\omega}| \leq \frac{S}{R} < 1$. We can interchange the sum with the integral:

  $$\int_{\mathcal{C}_R} \frac{f(\omega)}{\omega - z} d\omega = \sum_{n=0}^{+\infty} \int_{\mathcal{C}_R} \frac{f(\omega)}{\omega^{n+1}} d\omega z^n.$$

- For $\mathcal{C}_r$, consider $|\omega| = r$, and $|z| = s$ with $s > r$. Once again, using a geometric serie:

  $$\frac{1}{\omega - z} = \frac{1}{-z}\frac{1}{1 - \frac{\omega}{z}} = -\frac{1}{z}\sum_{n=0}^{+\infty} \frac{\omega^n}{z^n}.$$

  In fact, since $|\frac{\omega}{z}| \leq \frac{r}{s} < 1$, the convergence is uniform. We interchange the sum and the integral:

  $$\int_{\mathcal{C}_r} \frac{f(\omega)}{\omega - z} d\omega = -\sum_{n=0}^{\infty} \int_{\mathcal{C}_r} f(\omega)\omega^n d\omega \frac{1}{z^{n+1}}.$$

We have found the (a?) Laurent serie. To prove the uniqueness of the coefficients, consider $z = se^{i\theta}$, with $r < s < R$, and we get:

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n = \sum_{n\in\mathbb{Z}} a_n s^n e^{in\theta}.$$

For any $k \in \mathbb{Z}$, by uniform convergence, we get:

$$\int_{\mathcal{C}_s} f(z)e^{ik\theta}d\theta = \int_{\mathcal{C}_s} a_{-k-1}s^{-k-1}dz = 2\pi i a_{-k-1}s^{-k-1}.$$

We thus obtain the uniqueness of those coefficients. $\square$

For example, consider the function $f(z) = \frac{1}{z(z-1)}$ in the annulus $\mathcal{A}_{0,1}$. We have the decomposition:

$$f(z) = \frac{-1}{z} + \frac{1}{z+1} = \frac{-1}{z} - \frac{1}{1-z} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n.$$

We can also find a serie development in the annulus $\mathcal{A}_{1,R}$, for $R > 1$:

$$f(z) = -\frac{1}{z} + \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z} + \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z} = \sum_{n\geq 2}\frac{1}{z^n}.$$

Let $z_0 \in \mathbb{C}$, $r > 0$ and $f \in H(D'(z_0, r))$, where $D'(z_0, r) = D(z_0, r)_{\{z_0\}}$ (punctured disk). We denote its Laurent serie by:

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n.$$

In this case, z_0 is called a removable singularity. There are three cases of singularities.
The first case is a removable singularity: if for any $n < 0$, $a_n = 0$, then z_0 is a removable singularity. In other words, we can continuously extend f to the all disk, with $f(z_0) = a_0$. Furthermore, because it has an analytic development, the extension of f is holomorphic on the disk.
The second case is a pole of order m. If for any $n < -m$, where $m > 0$, we have $a_n = 0$ and $a_m \neq 0$, then $(z - z_0)^m f(z) \in H(D'(z_0, r))$ has a removable singularity at z_0, and can be continuously extended at z_0. z_0 is thus a pole of order m.

Definition 58. *Let U be an open set of $\mathbb{C}$, a finite set S of points $\{z_0, \cdots, z_n\}$., and $f \in H(U_{\setminus S})$. If each point of S is a pole of f, then f is called a meromorphic function.*

The third case is the one of essential singularity, and it occurs if an infinite number of $(a_n)_n$ are different from 0.

Theorem 59 (Casorati-Weierstrass)**.** *Let $f \in H(U_{\setminus\{z_0\}})$, $D'(z_0, r) \subset U$ and z_0 an essential singularity of f. Then $f(D'(z_0, r))$ is dense in $\mathbb{C}$.*

Proof. Suppose that it is false: there exists a certain $\omega \in \mathbb{C}$ and $\delta > 0$ such that $d(f(D'(z_0, r)), \omega) > \delta$. We can thus define the function on the punctured disk $g(z) := (f(z) - \omega)^{-1}$. g is holomorphic on the punctured disk, and bounded by $\frac{1}{\delta}$. We can write the Laurent serie on $D'(z_0, r)$:

$$g(z) = \sum_{n=-\infty}^{+\infty} a_n(z - z_0)^n, \quad \text{where } a_n = \frac{1}{2\pi i}\int_{\mathcal{C}_s} \frac{g(\omega)}{(\omega - z_0)^{n+1}}d\omega, \quad \forall 0 < s < r, \quad n < 0.$$

Because g is bounded on the punctured disk, a_n tends to O as s goes to 0. Thus $a_n = 0$, and g can be continuously extended at z_0 by 0. z_0 is a removable singularity of g, so the extension is in $H(D(z_0, s))$. Denote by N the smaller coefficient such that $a_N > 0$. Because the extension of g is analytic, $N \geq 0$; and because g is different from 0, N is finite. Thus:

$$g(z) = a_N(z - z_0)^N (1 + \tilde{g}(z)),$$

where $\tilde{g}(z_0) = 0$ and $\tilde{g}$ is holomorphic. Thus $\frac{1}{g}$ has at most one pole z_0, and is of finite order. So $f(z) - \omega$ has at most one pole, which is not an essential singularity. $\square$

Consider for example the function $f(z) = \frac{\sin(z)}{z}$. The singularity is a removable singularity, because:

$$\frac{\sin(z)}{z} = 1 + \sum_{n\geq 1}(-1)^n\frac{z^{2n}}{(2n+1)!} = 1 + h(z),$$

where $h \in H(\mathbb{C})$ and $h(0) = 0$. So for ϵ small enough, $|h| \leq \frac{1}{2}$ on $D(0, \epsilon)$, and f has a removable singularity at $z = 0$. The function can be extended at $z = 0$ by 1.

Now, consider the function $f(z) = \frac{1}{Log(z^2)}$ on $D'(1, \frac{1}{4})$ at 1, and Log is the principal value of the Logarithm. What is the singularity? We develop the series, with $\omega = z - 1$:

$$Log(z^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(z^2 - 1\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\omega^2 + 2\omega\right)^n = 2\omega + \omega^2 + h(\omega),$$

where $h(\omega) = O(\omega^2)$. Thus $Log(z^2) = 2(z-1) + \tilde{h}(z-1)$, and:

$$f(z) = \frac{1}{2(z-1)} + \frac{1}{2(z-1)} \sum_{n \geq 1} (-h(z-1))^n,$$

and 1 is a pole of order 1.
Take care, we cannot write without any justification $Log(z^2) = 2Log(z)$! It works well in this case where we are far from the negative axis, but it is not always the case. For example, with $z = e^{i\frac{3\pi}{4}}$, we have:

$$Log(z^2) = Log(e^{-i\frac{\pi}{2}}) = -i\frac{\pi}{2}, \quad \text{and } 2Log(e^{i\frac{3\pi}{4}}) = i\frac{3\pi}{2}.$$

Consider now the function $f(z) = \exp(\frac{1}{z})$. What is the nature of 0? You can use the development of the exponential. Another solution is the converse of the Casorati-Weierstrass: if we prove that for any neighbourhood small enough around 0, we can find an open set whose image is dense in $\mathbb{C}$, then the singularity is neither a removable singularity nor a pole. By the exponential, any strip $\{x + iy; n\pi < y < (n+2)\pi\}$ has a dense image in $\mathbb{C}$. Consider the preimage of the strip by $\frac{1}{z}$: we obtain a small domain, like an "earring", which is smaller when n increases. Thus 0 is an essential singularity.

Day 8

Definition 60. *An analytic automorphism of an open set U is an analytic function $f : U \to U$, which is isomorphic, and whose inverse is analytic.*

Theorem 61. *The only analytic automorphisms of $\mathbb{C}$ are the functions of the form : $f(z) = az + b$, where $(a, b) \in \mathbb{C}^2$ and $a \neq 0$. This set is denoted by $Aut(\mathbb{C})$.*

Proof. Let $f \in Aut(\mathbb{C})$. Then $g(z) := f(z) - f(z_0)$ is also an analytic automorphism, so we can focus on functions which are equal to 0 at 0. We need to prove that $f(z) = az$ with $a \neq 0$.
Let $f(z) = \sum_{n \geq 1} a_n z^n$, with $a_n \in \mathbb{C}$. Define the function $h(z) = f(\frac{1}{z}) = \sum_{n \geq 1} \frac{a_n}{z^n}$ on $\mathbb{C}^*$. This Laurent serie asks for the nature of the point 0 : is it an essential singularity? Consider $\epsilon > 0$. $f(D(0, \epsilon))$ is open (by the open mapping theorem) and contains a disk $D(0, \delta)$. Thus, for any $|\omega| > \epsilon$, we have $|f(\omega)| > \delta$. Furthermore, for any $|\omega| > \frac{1}{\epsilon}$, $|h(\omega)| > \delta$, and thus h is not dense in $\mathbb{C}$. 0 is thus not an essential singularity.
Because 0 is either a pole or a removable singularity of h, there is only a finite number of coefficients which are different from 0. f is a polynomial of degree $N < \infty$. By the D'Alembert-Gauss theorem, there is exactly N roots, counted with multiplicity. Because f is isomorphic, there is only one root, which is of multiplicity N : $f(z) = a_N z^N$. However, if $N > 1$, then $f(1) = f(e^{\frac{2i\pi}{N}})$, so it is not possible. Thus $N = 1$. ☐

3.5.1 The residue formula.

Suppose that the function f has the following Laurent serie at z_0:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n.$$

The residue of f at z_0 is the coefficient a_{-1} and is denoted by $Res_{z_0}(f) := a_{-1}$.

Theorem 62. *Let $z_0 \in \mathbb{C}$, and $f \in H(D'(z_0, R))$. Let $r < R$, and the circle $\mathcal{C}_r$ which is the range of γ:*

$$\begin{array}{rccc} \gamma : & [0, 2\pi] & \to & \mathbb{C} \\ & t & \mapsto & z_0 + re^{it} \end{array}. \tag{1}$$

Then:

$$Res_{z_0}(f) = \frac{1}{2\pi i} \int_{\mathbb{C}_r} f(\omega) d\omega.$$

Proof. Consider the compact annulus $\mathcal{A}_{\rho,\tilde{\rho}}$ with $0 < \rho < r < \tilde{\rho} < R$. Then the Laurent serie converges uniformly on $\mathcal{A}_{\rho,\tilde{\rho}}$ to f, and:

$$\frac{1}{2\pi i}\int_{\mathcal{C}_r} f(\omega)d\omega = \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}} a_n \int_{\mathcal{C}_r}(\omega - z_0)^n d\omega = \frac{1}{2\pi i}a_{-1}\int_{\mathbb{C}_r}\frac{1}{(\omega - z_0)}d\omega = Res_{z_0}(f).$$

□

Theorem 63 (Residue formula)**.** *Let U be an open set, γ a closed chain in U, such that $\gamma \sim 0$ (homologous) in U. Let $(z_0, \cdots, z_k) \subset U$, and $f \in H(U_{\setminus\{z_0,\cdots,z_k\}})$. Then:*

$$\int_\gamma f = 2\pi i \sum_{j=0}^{k} Ind_\gamma(z_j) Res_{z_j} f.$$

The proof is straightforward, since we have a finite number of points, and by considering a small curve γ_j around z_j as in (1), we get:

$$\gamma \sim Ind_\gamma(z_0)\cdot\gamma_0 \dotplus \cdots \dotplus Ind_\gamma(z_k)\cdot\gamma_k,$$

and the global Cauchy theorem.
This result is particularly useful when we deal with simply connected domain U.

For example, consider the function $f(z) = \frac{1}{\sin(z)}$. Does f have a pole at 0, and if it does, what is the residue? By a serie development, we get:

$$\sin(z) = \sum_{k=0}^{+\infty}\frac{(-1)^{2k+1}}{(2k+1)!}z^{2k+1} = z\left(1 + h(z)\right),$$

where $h(0) = 0$ and $|h(z)| \leq \frac{1}{2}$ in a small neighbourhood of 0, for example $D(0, \epsilon)$. We thus get:

$$\frac{1}{\sin(z)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-f(z)\right)^n,$$

with uniform convergence of the serie on the previous disk. Thus, 0 is a pole of order 1, and $Res_{z_0}(\frac{1}{\sin(z)}) = 1$.
Consider now the following example : what is the residue of f at $z = 1$, where:

$$f(z) = \frac{z}{z^3 - z^2 - z + 1}?$$

By noticing that $(z+1)(z-1)^2 = z^3 - z^2 - z + 1$, and with $y := z - 1$, we have:

$$\begin{aligned}
f(z) &= \frac{y+1}{y+2}\frac{1}{y^2} = \frac{y+1}{2y^2}\left(1 - \frac{y}{2} + \frac{y^2}{2} + O(y^3)\right) = \frac{1}{y^2}\left(\frac{1}{2} + \frac{y}{4} + O\left(y^2\right)\right)\\
&= \frac{1}{2y^2} + \frac{1}{4y} + O(1).
\end{aligned}$$

Thus $Res_1(f) = \frac{1}{4}$.

Theorem 64 (Argument principle, [1])**.** *Let U be an open set, and f be meromorphic in U, with zeros $\{z_0, \cdots, z_k\}$ with multiplicities $\{m_0, \cdots, m_k\}$ and poles $\{y_0, \cdots, y_l\}$ with orders $\{n_0, \cdots, n_l\}$. Then, for Γ a cycle homologous to 0 in U, and not passing through z_i nor y_i, we have:*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \sum_{i=0}^{k} m_i Ind_\Gamma(z_i) - \sum_{j=0}^{l} n_j Ind_\Gamma(y_j).$$

3.5.2 Extension to the Riemann sphere, [11, 9]

Let U be an open set of $\hat{\mathbb{C}}$, with $\infty \in U$, $0 \notin U$, and $f \in H(U_{\setminus\infty}, \mathbb{C})$. Let $g(z) := f\left(\frac{1}{z}\right)$. f has an isolated singularity at ∞ (resp. holomorphic, resp. meromorphic) if g has an isolated singularity at 0 (resp. holo, resp meromorphic). In particular, f is holomorphic at ∞ if g has a removable singularity at 0.

Theorem 65. *The only meromorphic functions on $\hat{\mathbb{C}}$ (thus with values in $\mathbb{C}$) are the rational functions. The only holomorphic functions on $\hat{\mathbb{C}}$ (thus with values in $\mathbb{C}$) are the oncstant functions. If f is holomorphic on $\mathbb{C}$ and has a pole at infinity, then f is polynomial.*

Proof. If f is meromorphic on $\hat{\mathbb{C}}$, then f has a finite number of zeros. Otherwise, we would obtain a converging subsequence of zeros in $\hat{\mathbb{C}}$, to a certain point $z_0 \in \hat{\mathbb{C}}$ (consider $|z_0| < \infty$ for simplicity). If z_0 is not a pole of f, then f is analytic at z_0, and by the accumulation point, $f = 0$. If z_0 is a pole of order m, the function $(z - z_0)^m f(z)$ is holomorphic on a neighbourhood of z_0, and thus the previous argument applies. Thus f has a finite number of zeros of finite multiplicity.
Consider now the function $g(z) = \frac{1}{f(z)}$, which is meromorphic on $\hat{\mathbb{C}}$, and the previous argument applied. g has a finite number of zeros of finite multiplicity, which corresponds to a finite number of poles of finite order of f. Let $\mathcal{Z}_{\mathbb{C}}(g) = \{y_0, \cdots, y_l\}$, with multiplicities $\{n_0, \cdots, n_l\}$. Consider now the function:

$$h(z) := f(z) \prod_{j=0}^{l} (z - y_j)^{n_j},$$

which can be extended as a holomorphic function on $\mathbb{C}$. Thus h is analytic on $\mathbb{C}$:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ with } R = \infty.$$

The function $h(\frac{1}{z}) = \sum_{n=0}^{+\infty} a_n z^{-n}$ is meromorphic at 0, because f is meromorphic at infinity. Thus $a_n = 0$ for any $n \geq N$, for N large enough. h is a polynomial, and f is a fractional function.

Day 9

For the second point, since $f \in H((\hat{\mathbb{C}}), \mathbb{C})$, and $\hat{\mathbb{C}}$ is compact, it implies that $f(\hat{\mathbb{C}}$ is compact in $\mathbb{C}$, thus bounded. The restriction of f to $\mathbb{C}$ is constant by the Liouville's theorem.
The last point uses the first one by noticing that $\mathcal{Z}_{\mathbb{C}}(\frac{1}{f}) = \emptyset$. □

We did not gave the true definition of meromorphic functions : we only consider functions which on open set have a finite number of points which are poles. In fact, we only need that the set of points where f has a pole to be discrete (without accumulation point in the set).

Definition 66. *Let U be an open set of $\mathbb{C}$. A function f is called meromorphic if f is holomorphic on $U \backslash S$, where S is a discrete set of points of U, and f has a pole at each point of S.*

For example, consider the meromorphic function on $\mathbb{C}^*$:

$$f(z) = \frac{1}{\sin(\frac{1}{z})}.$$

The poles of f are the zeros of $\sin(\frac{1}{z})$, thus for $z = \frac{1}{k\pi}$, with $k \in \mathbb{Z}^*$. Notice that $k = 0$ is not in the set. The set S has an accumulation point 0, but it is not in the domain $\mathbb{C}^*$. Notice that all the poles are of order 1.

Appendix : Logarithm of a function

Consider a region U (open and simply connected), and a function $f \in H(U, \mathbb{C})$, never equal to 0 on U. Can we define a function g such that $g = Log(f)$? In fact, this question is not well-posed, since the principal value of Logarithm is not defined on $\mathbb{C}$. It is better thus to ask the question : is there a function g satisfying $e^g = f$? Suppose that the answer is yes. We thus have:

$$f'(z) = g'(z) e^{g(z)}, \quad \frac{f'(z)}{f(z)} = g'(z).$$

We can integrate this formula, where z_0 is a point in U, and $\int_{z_0}^{z}$ represents the integral over a path from z_0 to z:

$$g(z) = \int_{z_0}^{z} \frac{f'(\omega)}{f(\omega)} d\omega + \ln |f(z_0)| + i\arg(f(z_0)).$$

The integral is well-defined, since U is simply connected and $\frac{f'}{f} \in H(U)$.

4 Conformal applications

This problem from cartographers: find a "map" from the Riemann sphere into a map (in the common sense, a plane in $2D$).
+ Make a picture
The problem is however, how to read the map, and to conserve certain quantities. Does is exist? In fact, we cannot conserve the distances (Reference?). For the area, I think it is possible (need of a reference also on it). If we exclude the north pole, there exists one map going from $S^2_{\setminus N}$ to $\mathbb{C}$ conserving the angles, and one example is... The stereographic projection! (we admit it, since to define an angle, we need to take the derivative of a path, and thus use the tangent space, not introduced in the course.)

4.1 The local point of view [11]

Definition 67. *Let U be an open set of $\mathbb{C}$, $f \in H(U)$. f is conformal at a point $z_0 \in \mathbb{C}$ if the angle made by two curves crossing at z_0 is preserved by f.*
f is conformal if f is conformal at z_0 for any $z_0 \in U$.

Example : consider the function $f(z) = \bar{z}$. This function conserves the angles, but not the orientation. In fact, the notion of a conformal application can be extended to non-holomorphic functions.
+pictures
What is the angle at z_0 made by γ and η, where:

$$\begin{cases} \gamma(t_0) = z_0, \\ \eta(t_1) = z_0, \end{cases} \qquad \text{with} \qquad \gamma'(t_0) \neq 0, \eta'(t_1) \neq 0?$$

It suffices to compute the angles:

$$\cos(\theta_{z_0}) = \left\langle \frac{\gamma'(t_0)}{|\gamma'(t_0)|}, \frac{\eta'(t_1)}{|\eta'(t_1)|} \right\rangle, \quad \sin(\theta_{z_0}) = \left\langle \frac{\gamma'(t_0)}{|\gamma'(t_0)|}, \frac{-i\eta'(t_1)}{|\eta'(t_1)|} \right\rangle,$$

where $\langle z, \omega \rangle = Re(z)Re(\omega) + Im(z)Im(\omega)$.
By computing the images of each path, we get: $(f \circ \gamma)'(t_0) = \gamma'(t_0) f'(\gamma(t_0))$ (the multiplication is the one complex numbers) and thus:

$$\begin{aligned} \left\langle (f \circ \gamma)'(t_0), (f \circ \eta)(t_1) \right\rangle &= \langle \gamma'(t_0) f'(\gamma(t_0)), \eta'(t_1) f'(\eta(t_1)) \rangle \\ &= |f'(\gamma(t_0))|^2 \langle \gamma'(t_0), \eta'(t_1) \rangle . \end{aligned}$$

The multiplication by $f'(\gamma(t_0)) = f'(\eta(t_1))$ is a composition of a rotation and a dilation, while $f'(\gamma(t_0)) \neq 0$.

Lemma 68. *f is conformal on U if $f' \neq 0$ for any point $z \in U$.*

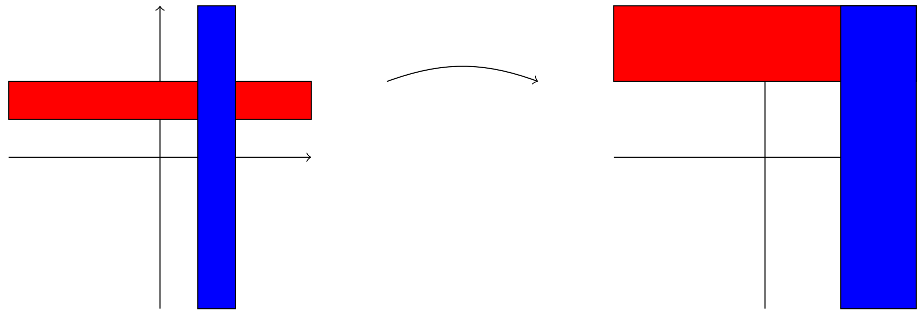
In particular, if $f : U \to V$ and $g : V \to W$ are conformal, then $g \circ f : U \to W$ is also conformal.
A natural question is to know either the converse of the lemma also holds : if f is conformal, is f never equal to 0? I need a reference for this proof, but here is my personal idea. Suppose that f is equal to 0 at $z_0 \in U$. If f is conformal, then the function $g(z) := f(z) - f(z_0)$ is also conformal (translation), and z_0 is a zero of multiplicity at least 2. By the theorem on the multiplicity of the zeros, by considering a small disk on $D(0, \delta)$, each equation $f(z) = \omega$ for $\omega \in D'(0, \delta)$ has at least 2 simple roots in $D(z_0, \epsilon)$. Maybe it is possible to construct a curve in $D(z_0, \epsilon)$ passing by z_0, and whose image is a path going to 0 and then goes backward on the same range.

4.2 Global point of view

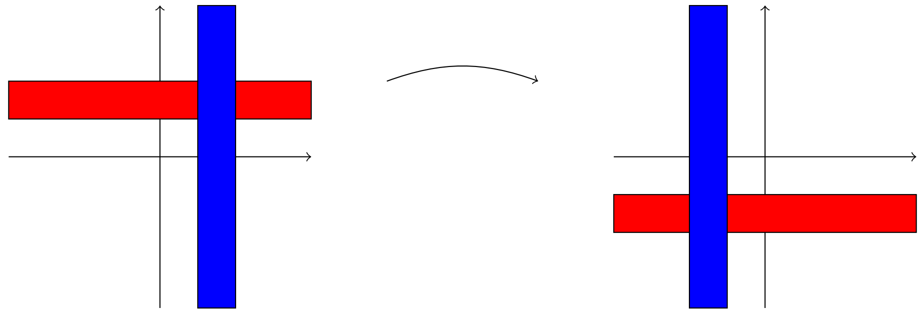
We already know some examples of functions (holomorphic). According to these functions, how will a certain region of the complex plane be transformed into?
In the rest of this section, the regions are open and connected.

4.2.1 Multiplication

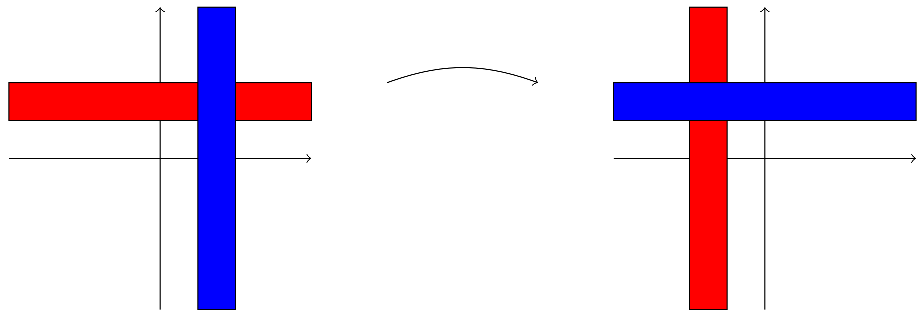
$z \mapsto az$, with $a \in \mathbb{R}^*$ (on the picture, $a > 1$: dilation):



$z \mapsto -z$ (point reflection; rotation of angle π):

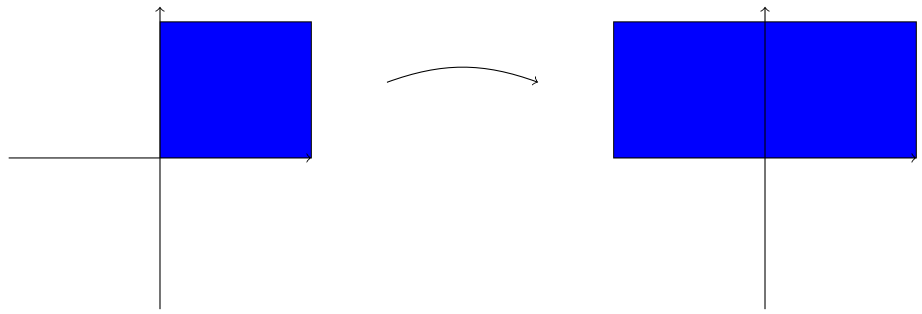


$z \mapsto iz$ (rotation of angle $\frac{\pi}{2}$):



4.2.2 Square function

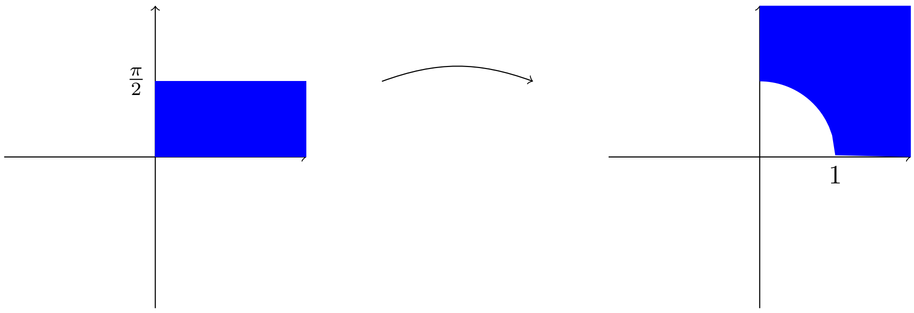
$z \mapsto z^2$



An interesting question is the non-injectivity of this function : is there any other domain in $\mathbb{C}$ whose range is the one on the right?

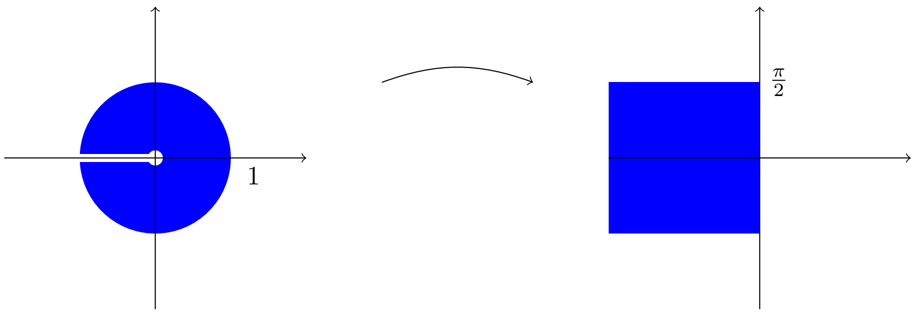
4.2.3 Exponential

$z \mapsto e^z$



4.2.4 Logarithm (principal value)

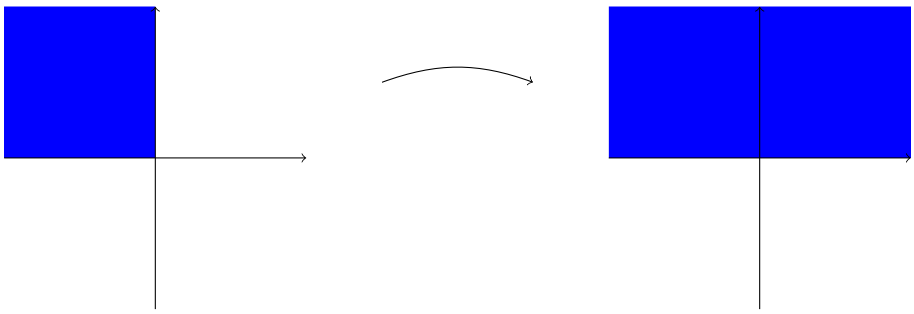
Log defined on $U = \mathbb{C} \backslash \mathbb{R}^-$.



$Log(re^{i\theta}) = r + i\theta$ for $-\pi < \theta < \pi$, $0 < r < 1$.
What is the image of the upper and lower intervals $[-1; 0]$? And of the point 0?

4.2.5 Square root

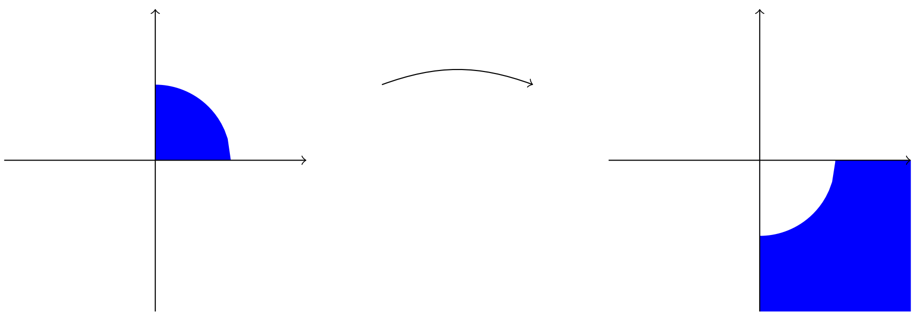
We define the function $f(z) = \exp\left(\frac{1}{2} Log(z)\right)$, defined on $\mathbb{C} \backslash \mathbb{R}^-$.



Write the intermediate steps with the logarithm and the exponential.
Make a comparison with the square function.

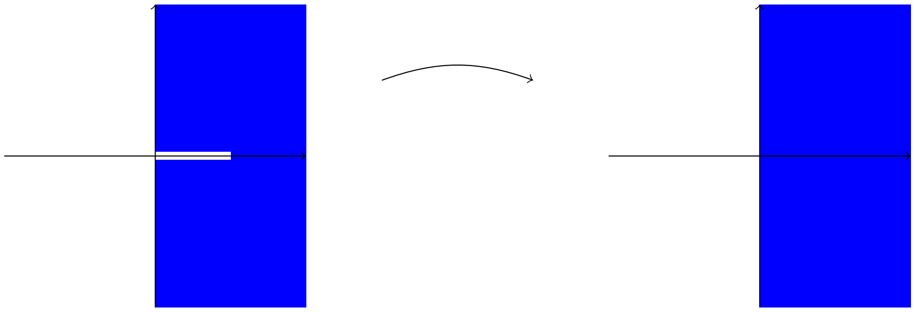
4.2.6 Inversion

$z \mapsto \frac{1}{z}$



4.2.7 Composition

$z \mapsto \sqrt{z^2-1}$, $U = \{z; Re(z) > 0; z \notin [0,1]\}$.



Notice that by the intermediate transformations, the boundary is conserved.

Day 10

4.2.8 From the upper plane to the circle [11]

Let $\mathbb{D}$ be the unit disk, and $\mathbb{H}$ be the upper half-plane.

Theorem 69. *The map $f : z \mapsto \frac{z-i}{z+i}$ sends $\mathbb{H}$ onto $\mathbb{D}$. Furthermore, it is bijective ; its inverse is also conformal.*

Proof. $\mathbb{H} = \{z; Im(z) > 0\}$. Let $z \in \mathbb{H}$. We have:

$$\begin{aligned}|f(z)| = \left|\frac{z-i}{z+i}\right| < 1 &\Leftrightarrow 0 < |z-i| < |z+i| \\ &\Leftrightarrow x^2 + (y-1)^2 < x^2 + (y+1)^2 \\ &\Leftrightarrow -2y < 2y\end{aligned}$$

which is satisfied if $y > 0$, thus f is well-defined.
For the inverse mapping:

$$\begin{aligned}\omega = \frac{z-i}{z+i} &\Leftrightarrow \omega\,(z+i) = z-i \\ &\Leftrightarrow z = -i\frac{\omega+1}{-1+\omega}.\end{aligned}$$

The map $h : \omega \mapsto -i\frac{\omega+1}{\omega-1}$ is well-defined from $\mathbb{D}$ into $\mathbb{H}$, since

$$Im(h(z)) = -Re\left(\frac{\omega+1}{\omega-1}\right) = \frac{-1}{|\omega-1|^2} Re\left((\omega+1)(\bar{\omega}-1)\right) > 0.$$

Thus f and h are injective, well-defined and we obtain the isomorphisms. □

4.3 Linear fractional transformations.

Definition 70. *Consider U a region of the Riemann sphere. The set of automorphisms of U, denoted by $Aut(U)$ is the of meromorphic bijections from U to U.*

For example, we proved that $Aut(\mathbb{C}) = \left\{z \mapsto az+b; (a,b) \in \mathbb{C}^2; a \neq 0\right\}$.
The following question is natural. We defined functions $f : U \to \mathbb{C}$, where U is a subset of $\mathbb{C}$. What does $f(z_0) = \infty$ mean?
If f is meromorphic on U, with a pole at $z_0 \in U$ of order $m > 0$ (thus $(z-z_0)^m f(z)$ is holomorphic at z_0), we denote $f(z_0) = \infty$ (it corresponds to zeros of $\frac{1}{f}$).

Theorem 71 ([9])**.**

$$Aut(\hat{\mathbb{C}}) = \left\{T : z \mapsto \frac{az+b}{cz+d}; ad-bc \neq 0,\ \text{with } a,b,c,d \in \mathbb{C}\right\}.$$

Those applications are called the linear fractional transformations (or Möbius transformations).

For example, the application $J : z \to \frac{1}{z}$ from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ is an automorphism. Its restriction to $\mathbb{C}^*$ is an automorphism of $\mathbb{C}^*$, and the two points 0 and ∞ are sent one onto the other:

$$\forall z \in D'(0,\epsilon), \quad J(z) = \frac{1}{z}, \text{ thus a pole of order 1 at 0;} \quad J(0) = \infty;$$
$$\forall z \in D'(0,\epsilon), \quad J(\frac{1}{z}) = z \underset{|z|\to 0}{\to} 0, \quad \text{thus } J(\infty) = 0.$$

Proof. Let $f \in Aut(\hat{\mathbb{C}})$.
First suppose, that $f(\infty) = \infty$. The restriction $f_{|\mathbb{C}}$ is in $Aut(\mathbb{C})$, and $f(z) = az + b$, $a \neq 0$.
Second, suppose that $f(z_0) = \infty$, with $z_0 \in \mathbb{C}$. z_0 is unique, since f is bijective. In particular, it is a pole of finite order m. Let $h(z) := (z - z_0)^m f(z)$ on $\mathbb{C}$. h is holomorphic on $\mathbb{C}$. Since it has a pole at ∞ ($h(\frac{1}{z}) = \frac{1}{z^m}(1 - zz_0)^m f(\frac{1}{z})$, with $|f(\infty)| < \infty$), h is a polynomial $P(z) = (z - z_0)^m f(z)$. We can suppose that P and $z - z_0$ are coprime, by dividing enough times by $z - z_0$. By this process, we still get $m \geq 1$, since $f(z_0) = \infty$ and $|P(z_0)| < \infty$. f has only one zero, so does P : $P(z) = a(z - z_1)^m$, with $z_1 \in \mathbb{C}$, $a \in \mathbb{C}$, $n \geq 1$, $z_1 \neq z_0$ (the case $z_1 = \infty$ is dealt with later). Consider the function $g(z) := a\frac{(z-z_1)^n}{(z-z_0)^m}$ on $D(z_1, \epsilon)$, with $z_0 \notin D(z_1, \epsilon)$. g is holomorphic, and by the theorem of mutliplicity of roots, there exists $\delta > 0$, such that for any $\omega \in D'(0, \delta)$, the equation $g(z) = \omega$ has exactly n simple roots in $D(0, \epsilon)$. Because f is injective, $n = 1$. We use the same argument on $l(z) = \frac{1}{f(z)}$ (since $z \mapsto \frac{1}{z} \in Aut(\hat{\mathbb{C}})$) to prove that $m = 1$. Thus $f(z) = a\frac{z-z_1}{z-z_0}$.
Finally, for the case $z_1 = \infty$, the function $(z - z_0)^m f(z)$ is a holomorphic function on $\mathbb{C}$, with a pole (or bounded) at infinity. It is thus a polynomial without 0 on $\mathbb{C}$, thus a constant, and $f(z) = \frac{a}{(z-z_0)^m}$. The fact that $m = 1$ follows from the previous step. □

Theorem 72 ([9]). *If $\mathbb{C}$ is a circle in $\hat{\mathbb{C}}$ (circle in $\mathbb{C}$ or a (line in $\mathbb{C}$) $\cup\, \infty$), and T a Möbius transformation, then $T(\mathbb{C})$ is a circle in $\hat{\mathbb{C}}$.*

Proof. Since T is:

$$T(z) = \frac{az + b}{cz + d} = \begin{cases} \frac{a}{c} + \frac{-ad+bc}{cz+d} & \text{if } c \neq 0, \\ \frac{a}{d}z + \frac{b}{d} & \text{if } c = 0, \end{cases}$$

it is a composition of the three following operations on $\hat{\mathbb{C}}$:

- $z \to z + a$, $a \in \mathbb{C}$ (restriction to $\mathbb{C}$: translation),
- $z \to az$, $a \neq 0$ (restriction to $\mathbb{C}$: dilation and rotation),
- $z \to \frac{1}{z}$ the inverse.

Each of the three operations keeps invariant the set of circles of $\hat{\mathbb{C}}$. For example, the circle $\mathcal{C}(z_0, r)$, with $0 \notin \mathcal{C}$, is sent by the image function on $\mathcal{C}(\frac{\bar{z_0}}{|z_0|^2 - r^2}, \frac{r^2}{(r^2 - |z_0|^2)^2})$ (to prove it, use the definition of the circle : $(z - z_0)\overline{z - z_0} = r^2$). Thus T also preserves the circles. □

Notice that a circle is defined by 3 points. In fact, a Möbius transformation is uniquely determined by the image of three distinct points:

Theorem 73 ([9]). *If z_1, z_2 and z_3 are three distinct points of $\hat{\mathbb{C}}$, then there exists a unique $T \in Aut(\hat{\mathbb{C}})$ such that $T(z_1) = 0$, $T(z_2) = 1$, $T(z_3) = \infty$.*

Corollary 74. *$Aut(\hat{\mathbb{C}})$ acts 3-transitively on $\hat{\mathbb{C}}$: for any $(z_1, z_2, z_3) \in \mathbb{C}^3$ and $(y_1, y_2, y_3) \in \mathbb{C}^3$, where all the $(z_i)_i$ (resp. $(y_i)_i$) are distinct, there exists $T \in Aut(\hat{\mathbb{C}})$ such that $T(z_i) = y_i$ for all i. Furthermore, this T is unique.*

Proof. (of the theorem) Let T be defined by:

$$T(z) = \frac{z - z_1}{z_1 - z_2}\frac{z_2 - z_3}{z_3 - z},$$

and if $z_i = \infty$, take the limit when $z_i \to \infty$. For example:

$$\lim_{z_2 \to \infty} T(z) = -\frac{z - z_1}{z_3 - z}.$$

Notice that the inverse of $T(z) = \frac{az+b}{cz+d}$ is

$$T^{-1}(\omega) = \frac{\omega d - b}{-\omega c + a}.$$

This T is unique; if U satisfies also the theorem:

$$UT^{-1}(0) = 0, \quad UT^{-1}(1) = 1, \quad UT^{-1}(\infty) = \infty.$$

By computing, the unique Möbius transformation letting 0, 1 and ∞ stable is the identity, thus $U = T$. $\square$

Proof. (of the corollary) Use $T : (z_1, z_2, z_3) \to (0, 1, \infty)$ and $U : (y_1, y_2, y_3) \to (0, 1, \infty)$, and consider $U^{-1} \circ T$. $\square$

Example : Let $T\mathbb{H} \to \mathbb{D}$. The boundary of the first one is sent on the boundary of the second one. Find T such that $T(0) = 1$, $T(1) = i$, $T(\infty) = -1$. By solving the system on a, b, c and d, we find $T(z) = \frac{z-i}{-z-i}$. This T sends the real line onto the unit circle, but we still need to check if $\mathbb{H}$ is sent in $\mathbb{D}$, and not $\bar{\mathbb{D}}^C$. Indeed, $T(i) = 0$, so it is fine.
In fact, if we would have changed $T(1) = -i$, we would have obtained $T : \mathbb{H} \to \bar{\mathbb{D}}^C$.

Remark 75. *From the point of view of projective geometry, the group* $Aut(\mathbb{C})$ *can be identified to* $PGL(2, \mathbb{C})$*, the group of homographies on* $\mathbb{P}^1(\mathbb{C})$*. A clear description of the group is made in [8].*

Remark 76. *The group of automorphisms of the Riemann sphere can also be described in terms of matrices, with* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\det(M) \neq 1$*, and the isomorphism keeps the law of composition. Studying the conservation of a certain subsets of geometric object can be brought to the study of a certain subgroup of matrices (for example on the upper half-plane in [11]).*

5 Harmonic functions

5.1 Real and complex derivations

The complex functions [18, 1] have been studied as depending on the variable z. Considered as functions on $\mathbb{R}^2$, we can write:

$$f(x+iy) = u(x,y) + iv(x,y),$$

where u and v are real-valued functions. Let us compute the derivative:

$$\frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad \text{when } h \text{ is real} \tag{2}$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = -i\frac{\partial f}{\partial y} \quad \text{when } h \text{ is imaginary.} \tag{3}$$

Consider now the function $g(z) := f(\bar{z})$, with $u_g(x,y) = u_f(x,-y)$ and $v_g(x,y) = v_f(x,y)$. We thus get:

$$\frac{\partial f(z)}{\partial \bar{z}} = \frac{\partial g(\bar{z})}{\partial \bar{z}} = \left(\frac{\partial g}{\partial z}\right)(\bar{z}) = \left(\frac{\partial u_g}{\partial x} + i\frac{\partial v_g}{\partial x}\right)(x,-iy) = \frac{\partial u_f}{\partial x} + i\frac{\partial v_f}{\partial x} = \frac{\partial f}{\partial x} \tag{4}$$

$$= \frac{1}{i}\left(\frac{\partial u_g}{\partial y} + i\frac{\partial v_g}{\partial y}\right)(x,-iy) = -\frac{1}{i}\frac{\partial u_f}{\partial y} + i\frac{\partial v_f}{\partial y} = i\frac{\partial f}{\partial y}. \tag{5}$$

It thus makes sense to take the derivative along z or $\bar{z}$. Indeed, by a change of variable $(z,\bar{z}) = (x+iy, x-iy)$ or $(x,y) = (\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2})$, it is coherent to define the derivatives by:

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

For example, consider the function $g(z,\bar{z}) = |z|^2 = z\bar{z}$. We compute the derivatives:

$$\partial_z g(z,\bar{z}) = \bar{z}, \quad \partial_{\bar{z}} g(z,\bar{z}) = z.$$

Definition 77. *Let $u : \mathbb{R}^2 \to \mathbb{R}$. u is (real-)differentiable at (x_0, y_0) if there exists a linear operator $T : \mathbb{R}^2 \to \mathbb{R}$ such that:*

$$\lim_{(h,k)\to 0 \text{ in } \mathbb{R}^2} \frac{|u(x+h,y+k) - u(x,y) - T(h,k)|}{|(h,k)|} = 0.$$

Proposition 78 ([18])**.** *Let f be continuous on $U \subset \mathbb{C}$, $f = u + iv$, with u and v real-differentiable. f is holomorphic if and only if u and v satisfy the Cauchy-Riemann equations.*

Proof. By a change of variable, it is equivalent to study $(u(x,y), v(x,y))$ or $\tilde{f}(z,\bar{z})$.
The direct sense is direct: f is holomorphic, so $\partial_x f = -i\partial_y f$ thus $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ and the equations are satisfied.
Conversely, if the Cauchy-Riemann equations are satisfied:

$$\frac{\partial \tilde{f}}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)(u+iv) = \frac{1}{2}(\partial_x u - \partial_y v) + \frac{i}{2}(\partial_y u + \partial_x v) = 0,$$

thus $\tilde{f}$ is independent of $\bar{z}$, and is holormorphic. □

If $\frac{\partial f}{\partial z} = 0$, f is called an anti-holomorphic function.

Definition 79. *Let Ω be an open set of $\mathbb{R}^2_{x,y}$. $u : \Omega \to \mathbb{R}$ is a harmonic function if f is continuous, $\partial_x^2 f$ and $\partial_y^2 f$ exist at every point, and $\Delta f := \partial_x^2 f + \partial_y^2 f = 0$. Δ is called the Laplacian.*
Similarly, let U be an open set of $\mathbb{C}$. $f : U \to \mathbb{C}$ is a harmonic function if f is continuous, $\partial_x^2 f$ and $\partial_y^2 f$ exist at every point, and $\Delta f := \partial_x^2 f + \partial_y^2 f = 0$.
In the complex case, it is equivalent to ask for the real part u and the imaginary part v to be harmonic.

Remark 80. *$\Delta t = 4\partial_z\partial_{\bar{z}} f(z,\bar{z})$. This operator appears naturally by studying some physical quantities in some contexts, such that the temperature, the potential functions, of the fluid flows (see [3, 11]).*

Theorem 81. *Holomorphic functions are harmonic.*

Do the proof yourself, it is easy.

Theorem 82. *If $u : \Omega \to \mathbb{R}$ is a harmonic function on a simply-connected region Ω of $\mathbb{R}^2$, then u is the real part of a holomorphic function.*

Since this result is very important and recurring, you will work on it during the seminars.
Example : the function $(x, y) \mapsto \ln(x^2 + y^2)$ is harmonic on an annulus which does not contain 0, but we cannot find a corresponding holomorphic function.

5.2 The Poisson kernel

Let's do a little aside (formal, but if you know the Fourier series, you can make the computations rigorous).
We want to find a solution f of the following problem : f is harmonic on the unit disk $\mathbb{D}$, and for g a function defined on the unit circle, we want $f(z) = g(z)$. Looking for a solution to this kind of problem is equivalent to look for a function $h(re^{i\theta}) = g(re^i\theta)$, where h satisfies on $\mathbb{D}$: $\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) h = 0$.
By the symmetry of the disk, we can hope for a decomposition of the form:

$$h(re^{i\theta}) = \sum_{n=0}^{\infty} a_n(\theta) r^n.$$

We thus obtain:

$$\frac{a_0''}{r} + \frac{a_1'' + a_1}{r} + \sum_{n=2}^{\infty} (n^2 a_n + a_n'') r^{n-2} = 0.$$

Thus:

$$\begin{aligned} a_0 &= \alpha_0 \theta + \beta_0, \\ a_1 &= \alpha_1 e^{i\theta} + \beta_1 e^{-i\theta} \forall n \geq 2, \quad a_n && = \alpha_n e^{in\theta} + \beta_n e^{-in\theta}. \end{aligned}$$

By the boundary conditions :

$$\alpha_0 \theta + \beta_0 + \sum_{n=1}^{\infty} \left(\alpha_n e^{in\theta} + \beta_n e^{-in\theta}\right) = g(\theta).$$

By periodicity, $\alpha_0 = 0$. Furthermore, for $m \neq n$,

$$\int_{\theta=0}^{2\pi} e^{im\theta} e^{in\theta} d\theta = \left[\frac{1}{i(m+n)} e^{i(m+n)\theta}\right]_0^{2\pi} = 0,$$

thus $\alpha_n = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-in\theta} d\theta$. We thus get a formula of h in terms of g:

$$\begin{aligned} h(re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} g(\phi) d\phi + \frac{1}{2\pi} \sum_{n=1}^{\infty} r^n \int_0^{2\pi} e^{-in\phi} e^{in\theta} g(\phi) + e^{in\phi} e^{-in\theta} g(\phi) d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\phi-\theta)} g(\phi) d\phi. \end{aligned}$$

Definition 83. *We define the Poisson kernel, for $0 \leq r \leq 1$ and t real, by:*

$$P_r(t) := \sum_{n=-\infty}^{+\infty} r^{|n|} e^{int}.$$

The sense of the sum on $\mathbb{Z}$ is the one of the previous mandatory assignment.

Proposition 84 ([18])**.** *The Poisson kernel satisfies the following properties:*

- *P_r is real, and for $z = re^{i\theta}$:*

$$P_r(\theta - t) = Re\left(\frac{e^{it} + z}{e^{it} - z}\right) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}.$$

- *for any $r \in [0;1)$, $\frac{1}{2\pi}\int_0^{2\pi} P_r(t)dt = 1$.*
- *$P_r(t) = P_r(-t)$. For any $0 < \delta < |t| < \pi$, $P_r(t) < P_r(\delta)$. For any $0 < \delta \leq \pi$, $\lim_{r\to 1} P_r(\delta) = 0$.*

The proof of those properties are interesting, you will work on it during the second mandatory assignment.

Day 12

Proposition 85 ([18])**.** *If $\int_0^{2\pi} |g(e^{i\phi})d\phi < \infty$, then*

$$f(re^{i\theta}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi)g(e^{i\phi})d\phi$$

is a harmonic function in $\mathbb{D}$. Furthermore, the function $Hf(z) = f(z)$ on $\mathbb{D}$ and $Hf(z) = g(z)$ on $\mathcal{C}(0,1)$, if g is continuous, is also continuous.

Proof. By separating the real and imaginary parts:

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(P_r(\theta-\phi)g(e^{i\phi})\right)d\theta + \frac{i}{2\pi}\int_{-\pi}^{\pi}\mathrm{Im}\left(P_r(\theta-\phi)g(e^{i\phi})\right)d\theta.$$

We can thus differentiate inside the integrals (seen as real integrals, with the dominated convergence theorem). We thus get:

$$\Delta f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)P_r(\theta-\phi)\,g(e^{i\phi})d\phi$$
$$\Delta P_r(\theta-\phi) = \partial_z\partial_{\bar z}\mathrm{Re}\left(\frac{e^{i\phi}+z}{e^{i\phi}-z}\right) = \partial_z\partial_{\bar z}\left(\frac{1}{2}\frac{e^{i\phi}+z}{e^{i\phi}-z} + \frac{1}{2}\frac{e^{-i\phi}+\bar z}{e^{i\phi}-\bar z}\right) = 0.$$

f is thus harmonic on $\mathbb{D}(0,1)$. To prove the continuity when $|z|\to 1$, we use the density of trigonometric functions in $\mathcal{C}([0;2\pi])$. If $\tilde g$ satisfies:

$$\tilde g(e^{i\phi}) = \sum_{|l|\leq N} a_l e^{il\phi}, \quad \|g(e^{i\cdot}) - \tilde g(e^{i\cdot})\|_{\mathcal{C}([0,1])} \leq \epsilon,$$

we have, for $r<1$:

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi)g(e^{i\phi})d\phi - g(e^{i\theta})\right| \\
&\quad\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi)\|g(e^{i\cdot})-\tilde g(e^{i\cdot})\|_{\mathcal{C}([0,1])}d\phi + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} r^{|n|}\sum_{|l|\leq N} a_l e^{i(l\phi+n(\phi-\theta)}d\phi - g(e^{i\theta})\right| \\
&\quad\leq \epsilon + \left|\sum_{|l|\leq N}\frac{1}{2\pi}\int_{-\pi}^{\pi} r^{|l|}a_l e^{il\theta}d\phi - g(e^{i\theta})\right| \underset{r\to 1}{\to} \epsilon.
\end{aligned}$$

Since this inequality is true for any ϵ, we obtain the continuity of Hf on $\overline{\mathbb{D}(0,1)}$. ☐

Remark 86. *The solution of the problem:*

$$\begin{cases}\Delta f = 0, \text{ on } \mathbb{D},\\ f = g, \text{ on } \partial\mathbb{D},\end{cases} \tag{6}$$

for g continuous and f continuous on $\mathbb{D}(0,1)$ is unique (we need to work a bit more).

For example, consider the function $g(e^{i\theta}) = \cos(2\theta)$, we can solve the Dirichlet problem by developing the function $g:\, g(e^{i\theta}) = \cos(2\theta) - \sin(2\theta)$, and thus $f(x,y) = x^2 - y^2$, or in complex variables, $f(z,\bar z) = \frac{z^2}{2} + \frac{\bar z^2}{2}$.

Theorem 87 ([11])**.** *Let U be open in $\mathbb{C}$ (or in $\mathbb{R}^2$), and u harmonic on U, $u \in \mathcal{C}(\overline{U})$, u real-valued. For any $z \in U$, consider $R_z > 0$ such that $D(z,R_z) \subset U$. Then $\forall r < R_z$, we have:*

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})dt = u(z).$$

Proof. u is the real part of a certain holomorphic function f. We thus have (Cauchy formula):

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\omega)}{\omega - z} d\omega,$$

on the circle centered at z and of radius r.

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi i} \frac{f(z+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta,$$

and we separate the real and imaginary parts. □

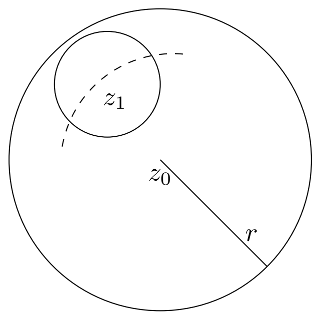
Notice that there exists a proof in [18] using the property of the Poisson kernel.

5.3 Main theorems on harmonic functions

Corollary 88 (Maximum principle)**.** *1. Let u be harmonic on a region U. If U has a maximum at a point $z_0 \in U$, then u is constant.*

2. Consider U a bounded region and $u \in \mathcal{C}(\overline{U}, \mathbb{R})$ a harmonic function on U. If u is not constant on U, then a maximum of u on $\overline{U}$ occurs on the boundary ∂U.

Proof. Let us prove the first point. Suppose that the maximum is achieved at a point $z_0 \in U$. Consider a disk $D(z_0, r) \subset U$. If on this disk, there exists z_1 with $u(z_1) < u(z_0)$, then there exists a disk $D(z_1, \rho) \subset D(z_0, r)$ where $u(z) < u(z_0)$ on $D(z_1, \rho)$ (by continuity).



Let $\tilde{\rho} := |z_0 - z_1|$, and an interval I of $[0, 2\pi]$ such that:

$$\forall \theta \in I, \quad z_0 + \tilde{\rho}e^{i\theta} \in D(z_1, \rho).$$

Thus:

$$\begin{aligned} u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \tilde{\rho}e^{i\theta})d\theta &= \frac{1}{2\pi}\int_I u(z_0 + \tilde{\rho}e^{i\theta})d\theta + \frac{1}{2\pi}\int_{[0,2\pi]\setminus I} u(z_0 + \tilde{\rho}e^{i\theta})d\theta \\ &< \frac{1}{2\pi}\int_I u(z_0)d\theta + \frac{1}{2\pi}\int_{[0,2\pi]\setminus I} u(z_0)d\theta = u(z_0). \end{aligned}$$

This is a contradiction, and u is constant on the disk.
For the second part of the proof, we use the local harmonic (or holomorphic) property to deduce a global property. Let z_0 be a point where the maximum is achieved. Consider a point $z \in U$, and a simply-connected region V containing z and z_0. u is the real part of a holomorphic function f on V:

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ \forall z \in D(z_0, r), \quad \partial_z f(z) &= 0 + i\partial_z v(z) = 0, \end{aligned}$$

since by the Cauchy-Riemann equation, $\partial_x v = \partial_y v = 0$. Thus f is constant on $D(z_0, r) \subset V$. Because $D(z_0, r)$ is not discrete in V, $f = C$ on $f = C$ on V. Thus $u = Re(C)$ on V, and thus on U.
The second part of the corollary is a consequence. □

Remark 89. *[11] uses for this proof the maximum principle of holomorphic functions.*

Theorem 90 (Inverse mean value theorem)**.** *Let U be open in $\mathbb{C}$, u continuous on U and real valued. Suppose that:*

$$\forall z \in U, \exists R > 0, \forall r \in (0, R), \quad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})dt.$$

Then u is harmonic.

Proof. Let us fix $\overline{D(z_0, R)} \subset U$, and

$$u_1(z_0 + re^{i\theta}) := \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(z_0 + re^{i\phi}) d\phi,$$

and $h = u_1 - u$. Let $m := \sup \left\{ h(z); z \in \overline{D(z_0, R)} \right\}$.
Assume $m > 0$. Since u is continuous on U, u_1 is continuous on the closed disk. In particular, by the previous theorem, $h = 0$ on the boundary of the disk. Let $E := \left\{ z \in \overline{D(z_0, R)}; h(z) = m \right\}$; by the previous remark, $E \subset D(z_0, R)$. Because h is continuous on the disk, E is closed and thus compact.
There exists a point $z_1 \in E$ such that $|z_1 - z_0| \geq |z - z_0|$ for any $z \in E$. Let $r < d(z_1, \partial D(z_0, R))$. Then for more than half of a circle $D(z_1, r)$, we have $u(z) < m$, and $\frac{1}{2\pi} \int_0^{2\pi} u(z_1 + re^{it}) dt < u(z_1)$, which is aburd.
Thus E is empty, and $m \leq 0$. The same proof holds for $m \geq 0$, and $m = 0$ implies that u is locally defined by the Poisson formula, which gives a harmonic function. □

We denote by $\mathbb{H}^+$ the upper half-plane, and $\mathbb{H}^-$ the lower half-plane.

Theorem 91 (Schwarz reflection principle, [11]). *Let I be a real open interval. Let U^+ be open in $\mathbb{H}^+$, such that for any $x \in I$, x is the center of a disk D_x which satisfies:*

$$D_x \cap \mathbb{H}^+ \subset U^+.$$

Let U^- be the reflection of U^+ with respect to the real axis :

$$U^- := \left\{ z \in \mathbb{C}; \bar{z} \in U^+ \right\}.$$

Let $v : U^+ \cup I \to \mathbb{R}$ be a harmonic function on U^+, and continuous on $U^+ \cup I$, with $v(x) = 0$, for any $x \in I$. The function v extends to a harmonic function on $U^+ \cup I \cup U^-$.

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Proof. We define v on U^- by $v(z) = -v(\bar{z})$.

- v is continuous on $U^+ \cup I \cup U^-$ (by symmetry).
- v is harmonic on U^-:

$$\Delta v(z) = \left(\partial_x^2 + \partial_y^2\right)(v(x, -y)) = (\Delta v)(x, -y) = 0.$$

- v is harmonic on $U^+ \cup I \cup U^-$. Indeed, for $x \in I$ and $r > 0$ small enough, $D(x, r) \subset U^+ \cup I \cup U^-$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} v(x + re^{it}) dt = \frac{1}{2\pi} \int_0^{\pi} v(x + re^{it}) dt + \frac{1}{2\pi} \int_0^{\pi} v(x + re^{-it}) dt = 0,$$

  since $v(x + re^{-it}) = v(\overline{x + re^{it}}) = -v(x + re^{it})$. Thus $v(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(x + re^{it}) dt = 0$, and we conclude by the inverse mean value theorem.

□

Remark 92. *This theorem has also an equivalent for holomorphic function on U^+.*

Remark 93. *We can use the Riemann mapping theorem to obtain this result with a cirlce instead of a line.*

Corollary 94 ([1]). *If $f \in H(U^+)$, f continuous on $U^+ \cup I$, and f real on I, then f has a holomorphic extension on $U^+ \cup I \cup U^-$.*

Proof. On $z \in U^-$, we define:

$$f(z) := \overline{f(\bar{z})} = u(x, -y) - iv(x, -y),$$
$$\operatorname{Im}(f(z)) = -Im(f(\bar{z})).$$

v satisfies the previous theorem. □

6 Riemann mapping theorem

6.1 Conformally equivalent regions [18]

We have studied conformal applications.

Definition 95. *[18] Two regions U_1 and U_2 are conformally equivalent if there exists $\phi \in H(U_1)$ one-to-one in U_1 and such that $\phi(U_1) = U_2$.*

We can prove that at any point, $\phi'(z) \neq 0$. If there exists $z_0 \in U_1$ such that $\phi'(z_0) = 0$, let $\omega_0 := \phi(z_0)$, and $V_2 := U_2 - \omega_0$. The function $\psi : z \mapsto \phi(z) - \omega_0$ is one-to-one from U_1 onto V_2. In particular, $\psi(z_0) = \psi'(z_0) = 0$, and by the theorem on the multiplicity of zeros, there exists $\omega \in D(\omega_0, r)$ for r small enough, such that the equation $\phi(z) = \omega$ has at least two solution in U_1. Thus for any $z \in U_1$, $\phi'(z) \neq 0$.

Remark 96. *$f \mapsto f \circ \phi$ is a mapping from $H(U_2)$ to $H(U_1)$. Since ϕ is invertible, so is the previous mapping. Studying a problem on $H(U_2)$ is equivalent to study the problem on $H(U_1)$.*

Undoubtely, we reduce the study to one the easiest and well-known case, the circle:

Theorem 97 (Riemann mapping theorem)**.** *Every simply-connected region U in the complex plane $\mathbb{C}$ (except $\mathbb{C}$ itself) is conformally equivalent to the open unit disk $\mathbb{D}$.*

It is better to know the mapping when dealing with a particular geometry!

6.2 Normal families [18]

Let X be a subset of $\mathbb{C}$.

Definition 98. *[18] Let $\mathcal{F}$ be a family of continuous function of $\mathbb{C}(X, \mathbb{C})$. $\mathcal{F}$ is equicontinuous if:*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in U, \forall f \in \mathcal{F}, \quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

It implies in particular that each f is uniformly continuous.

Theorem 99 (Arzelà-Ascoli)**.** *If $\mathcal{F}$ is a pointwise bounded equicontinuous family of $\mathcal{C}(K, \mathbb{C})$, and $\mathcal{F}$ is at least countable ($> \mathbb{N}$), where K is a compact set. Every sequence $(f_n)_n \subset \mathcal{F}$ has a subsequence which converges uniformly on K.*

For example, consider the functions $f_n(z) = z^n$ on $U = \mathbb{D}$. The family of the functions restricted to a compact set has a subsequence which uniformly converges to 0 on K, but no subsequence converges uniformly to 0 on the open unit disk.

Definition 100. *Let $\mathcal{F}$ be a family of $H(U)$, for some region U. $\mathcal{F}$ is a normal family if every sequence of members of $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of U.*

Remark 101. *The family $\mathcal{F} = \{f_n; n \in \mathbb{N}\}$ composed of functions previously defined is normal. The sequences converge uniformly on any compact set to 0. Thus we do not need that the limit is in the family $\mathcal{F}$.*

Theorem 102. *Montel Let $\mathcal{F} \subset H(U)$ and $\mathcal{F}$ is uniformly bounded on each compact subset of the region U. Then $\mathcal{F}$ is a normal family.*

Proof. Let us consider the compact sets:

$$K_n := D(0, n) \cap \left\{ x \in U; d(x, \partial U) \geq \frac{1}{n} \right\}.$$

The sequence $(K_n)_n$ increases (in the sense of the inclusion), and satisfies:

$$U = \bigcup_n K_n; \quad \partial K_n \subset \overset{\circ}{K}_{n+1}; \quad \forall n, \forall x \in K_n, D(x, \delta_n) \subset K_{n+1},$$

where $\delta_n = -\frac{1}{n+1} + \frac{1}{n}$. Thus each compact K is in K_n for n large enough. It suffices to prove the uniform convergence on any K_n.
Let us fix n. There exists a constant $M(K_n) > 0$, such that:

$$\forall f \in \mathcal{F}, \forall x \in K_n, \quad |f(x)| \leq M(K_n).$$

The family $\mathcal{F}$ is equicontinuous on K_n. Let z, z' such that $|z-z'| \leq \frac{\delta_n}{2}$. Let $\gamma : t \mapsto z + \delta_n e^{it}$ be a closed curve with value in K_{n+1}. z' is in the disk of boundary γ^*. We have:

$$f(z) - f(z') = \frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega - z} - \frac{f(\omega)}{\omega - z'} d\omega = \frac{z-z'}{2i\pi}\int_\gamma \frac{f(\omega)}{(\omega-z)(\omega-z')}d\omega,$$

$$|f(z) - f(z')| \leq \frac{|z-z'|}{\delta_n^2} M(K_{n+1}) 2\delta_n.$$

Thus for any n, the restrictions of $f \in \mathcal{F}$ to K_n is an equicontinuous family. Consider $(f_m)_m \subset \mathcal{F}$. For $n = 1$, we can extract $\left(f_{\phi_1(m)}\right)_m$ such that (Arzelà-Ascoli) the sequence converges uniformly on K_1. We then extract $(f_{\phi_1 \circ \phi_2(m)})_m$ of $(f_{\phi_1(m)})_m$ so that the convergence holds on K_2. By a diagonal process, $\left(f_{\phi_1 \circ \cdot \circ \phi_m(m)}\right)_m$ converges uniformly on each compact K_n, thus on each compact of U. □

6.3 Application to the Dirichlet problem

Let U be a region, and a function $g \in \mathcal{C}(\partial U, \mathbb{R})$. The Dirichlet problem consists in finding a solution f of :

$$\begin{cases}\Delta f = 0 \quad \text{on } U, f = g \qquad \text{on } \partial U.\end{cases} \tag{7}$$

(This definition is up to modification according to the context, since the same problem can occur in other contexts. For example, the boundary of U may not be regular, it can also be a manifold. It means that the Laplacian needs also another definition. The function g can also be considered with more or less regularity.) Consider a simply-connected region U. By the Riemann mapping theorem, consider the mapping $\phi : \bar{U} \to \bar{\mathbb{D}}$ which makes U and $\mathbb{D}$ conformally equivalent. If we find solution $\tilde{f}$ of:

$$\begin{cases} \Delta \tilde{f} = 0 & \text{on } \mathbb{D}, \\ \tilde{f} = g \circ \phi^{-1} & \text{on } \partial\mathbb{D}, \end{cases} \tag{8}$$

then $\tilde{f} \circ \phi$ solves (7).

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To prove it, we denote:

$$\phi : \begin{array}{ccc} \bar{U} & \to & \bar{\mathbb{D}} \\ (x,y) & \mapsto & \phi(x,y) = (u(x,y), v(x,y)) \end{array}, \qquad \tilde{f} : \begin{array}{ccc} \mathbb{D} & \to & \mathbb{C} \text{ or } \mathbb{R} \\ (u,v) & \mapsto & f(u,v). \end{array}$$

We can compute the derivative:

$$\begin{aligned}
\partial_x \left(\tilde{f} \circ \phi\right) &= \partial_u \tilde{f} \partial_x u + \partial_v \tilde{f} \partial_x v \\
\partial_x^2 \left(\tilde{f} \circ \phi\right) &= \partial_u^2 \tilde{f} \left(\partial_x u\right)^2 + \partial_u \tilde{f} \partial_x^2 u + 2\partial_u \partial_v \tilde{f} \partial_x u \partial_x v + \partial_v^2 \tilde{f} \left(\partial_x v\right)^2 + \partial_v \tilde{f} \partial_x^2 v \\
\partial_y^2 \left(\tilde{f} \circ \phi\right) &= \partial_u^2 \tilde{f} \left(\partial_y u\right)^2 + \partial_u \tilde{f} \partial_y^2 u + 2\partial_u \partial_v \tilde{f} \partial_y u \partial_y v + \partial_v^2 \tilde{f} \left(\partial_y v\right)^2 + \partial_v \tilde{f} \partial_y^2 v \\
\Delta \left(\tilde{f} \circ \phi\right) &= \Delta \tilde{f} \left(\left(\partial_x u\right)^2 + \left(\partial_y v\right)^2\right) + 2\partial_u \partial_v \tilde{f} \left(\partial_x u \partial_x v + \partial_y u \partial_y v\right) + \partial_u \tilde{f} \Delta u + \partial_v \tilde{f} \Delta v.
\end{aligned}$$

This sum is equal to 0 since $\tilde{f}$ is harmonic, and ϕ satisfies the Cauchy-Riemann equations.

6.4 Automorphisms of the unit disk $\mathbb{D}$

Lemma 103 (Schwarz, see exercises)**.** *Let $f : \mathbb{D} \to \mathbb{D}$, $f \in H(\mathbb{D})$, with $f(0) = 0$.*

- *We have $|f(z)| \leq |z|$, for any $z \in \mathbb{D}$.*
- *If for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$, then f is a rotation : there exists $|\alpha| = 1$, such that $f(z) = \alpha z$.*
- *Let $f(z) = a_1 z$+higher terms. Then $|f'(0)| = |a_1| \leq 1$, and if $|a_1| = 1$, then $f(z) = a_1 z$.*

The goal is to bring automorphisms to function with $f(0) = 0$. To do so, for $\alpha \in \mathbb{D}$, we denote the following Möbius transformation:

$$g_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Since $|\alpha| \neq 1$, we have $\alpha\bar{\alpha} + (1)(-1) \neq 0$.
If $z = e^{i\theta} \in \partial\mathbb{D}$, we have:

$$\begin{aligned} g_\alpha(z) &= \frac{\alpha - e^{i\theta}}{e^{i\theta}\left(e^{-i\theta} - \bar{\alpha}\right)} = \frac{-1}{e^{i\theta}} \frac{\alpha - e^{i\theta}}{\overline{\alpha - e^{i\theta}}}, \\ |g_\alpha(z)| &= 1. \end{aligned}$$

Thus the unit circle is invariant by g_α. If we compute g_α^2:

$$g_\alpha \circ g_\alpha(z) = \frac{\alpha - \frac{\alpha - z}{1 - \bar{\alpha}z}}{1 - \bar{\alpha}\frac{\alpha - z}{1 - \bar{\alpha}z}} = \frac{\alpha(1 - \bar{\alpha}z) - \alpha + z}{1 - \bar{\alpha}z - \bar{\alpha}\alpha + \bar{\alpha}z} = \frac{z(1 - \alpha\bar{\alpha})}{1 - \bar{\alpha}\alpha} = id(z).$$

Theorem 104 ([11])**.**

$$Aut(\mathbb{D}) = \left\{f; \exists|\alpha| < 1, \exists\phi \in \mathbb{R}, f = e^{i\phi}g_\alpha\right\}.$$

Furthermore, if $f \in Aut(\mathbb{D})$, and for $\alpha \in \mathbb{D}$, $f(\alpha) = 0$, then there exists $\phi \in \mathbb{R}$ such that:

$$f = e^{i\phi}g_\alpha.$$

Proof. Let $f \in Aut(\mathbb{D})$, and $f(\alpha) = 0$, with $\alpha \in \mathbb{D}$. Thus $f \circ g_\alpha \in Aut(\mathbb{D})$, with $f \circ g_\alpha(0) = 0$.
By the Schwarz lemma (second point): $|f \circ g_\alpha(z)| \leq |z|$ for any $z \in \mathbb{D}$.
Since $f \circ g_\alpha \in Aut(\mathbb{D})$, we have:

$$|(f \circ g_\alpha)^{-1}(z) \leq |z| \quad \Leftrightarrow \quad |\omega| \leq |f \circ g_\alpha(\omega)|, \quad \forall\omega \in \mathbb{D}.$$

Thus, for any $z \in \mathbb{D}$, $|f \circ g_\alpha(z)| = |z|$, and by the third point of the Scharz lemma, there exists $\phi \in \mathbb{R}$, such that $f \circ g_\alpha(z) = e^{i\phi}z$, or in other words, $f(\omega) = e^{i\phi}g_\alpha(\omega)$. □

6.5 Proof of the Riemann mapping theorem [18]

Theorem 105. *Every simply-connected region U in $\mathbb{C}$, with $U \neq \mathbb{C}$, is conformally equivalent to $\mathbb{D}$.*

Proof. Consider the set:

$$\Sigma := \{\Psi \in H(U); \Psi(U) \subset \mathbb{D}, \text{ and } \Psi \text{ injective }\}.$$

First step Σ is not empty. Since U is simply connected, and $U \neq \mathbb{C}$, consider $z_0 \notin U$, and $z_1 \in U$. Consider a determination of the logartihm:

$$\log(z - z_0) = g(z) := \int_{z_1}^{z} \frac{1}{\omega - z_0} dw.$$

This function is well defined, since the function $\omega \mapsto \frac{1}{\omega - z_0}$ is holomorphic on U.
g is injective, since g satisfies, for a certain constant A, $e^{g(z)} = A(z - z_0)$ (the derivative of $z \mapsto e^{g(z)}(z - z_0)^{-1}$ is null), and thus:

$$g(z_1) = g(z_2) \quad \Rightarrow \quad e^{g(z_1)} = e^{g(z_2)} \quad \Rightarrow \quad z_1 = z_2.$$

In fact, with the same argument, we can prove $g(z_0) + 2\pi i \notin Im(g)$. However, by the open mapping theorem, there exists $\delta > 0$ such that $D(g(z_0), \delta) \subset Im(g)$. So it implies, with the same argument, that
$D(g(z_0) + 2\pi, \delta) \cap Im(g) = \emptyset$.
Thus the application:

$$f : z \mapsto \frac{\delta}{g(z) - g(z_0) - 2\pi i}$$

is well-defined from U to $\mathbb{D}$. Thus $f \in \Sigma$.
Second step Let $f \in \Sigma$. We prove that if f does not cover $\mathbb{D}$, and $z_0 \in U$, we can find $h \in \Sigma$ such that:

$$|h'(z_0)| > |f'(z_0)|.$$

Here is my own intuition : if we define a non-surjective mapping from U to $\mathbb{D}$, then we can "stretch" the function f in some direction to obtain a bigger image set.

Let $\alpha \in \mathbb{D}$, $\alpha \notin f(U)$. Then $g_\alpha \circ f \in \Sigma$, without zero on U (recall that $g_\alpha(z) = \frac{-z+\alpha}{1-\bar{\alpha}z}$). $g_\alpha \circ f(U)$ is a simply connected region in $\mathbb{D}$.
We can define a determination of a square root of $g_\alpha \circ f$:

$$\sqrt{g_\alpha \circ f} := \exp\left(\frac{1}{2}\log(g_\alpha \circ f)\right),$$

where the log is one determination (we have seen before how to define a determination).
Let $\beta := \sqrt{g_\alpha \circ f}(z_0)$. Then the application:

$$h := g_\beta\left(\sqrt{g_\alpha \circ f}\right)$$

is injective (since each map is injective); defined from U to $\mathbb{D}$ (the square-root determination sends a point of the unit disk onto the unit disk); analytic; $h(z_0) = 0$. Thus this new function is in Σ.
Consider now the function $F := g_{-\alpha} \circ s \circ g_{-\beta} = g_\alpha \circ s \circ g_\beta$, where $s(\omega) = \omega^2$. We have:

$$\begin{aligned} f &= F \circ h \\ f'(z_0) &= F'(h(z_0))h'(z_0) = F'(0)h'(z_0). \end{aligned}$$

Since $F(\mathbb{D}) = \mathbb{D}$, and F is not bijective. Thus F satisfies (third step) $|F'(0)| < 1$, and h satisfies the expected property.
Third step The previous function F satisfies $|F'(0)| < 1$. Let $\omega_0 = F(0) \in \mathbb{D}$. Then $(g_{\omega_0} \circ F) \in H(\mathbb{D}, \mathbb{D})$, and $g_{\omega_0} \circ F(0) = 0$. We can compute the derivative at the origin, and by the third point of the Scharz lemma ($g_{\omega_0} \circ F$ is not surjective):

$$|\left(g_{\omega_0} \circ F\right)'(0)| = |F'(0)g'_{\omega_0}(\omega_0)| < 1.$$

Since $g'_{\omega_0}(\omega_0) = -\frac{1}{1-|\omega_0|^2}$, we have $|g'_{\omega_0}(\omega_0)| \geq 1$, and $|F'(0)| < 1$.
Fourth step Let $z_0 \in U$, and:

$$\eta := \sup\left\{|f'(z_0)|; f \in \Sigma\right\}.$$

By the previous point, it suffices to prove the existence of $h \in \Sigma$ such that $|h'(z_0)| = \eta$ to finish.
Since for any $f \in \Sigma$, $z \in U$, $|f(z)| < 1$, Σ is a normal family (by Montel theorem). By definition of η, there exists $(f_n)_n \subset \Sigma$ such that $|f'_n(z_0)| \to \eta$; we can then extract a subsequence (denoted $(f_n)_n$) which converges to $h \in H(U)$.
By the convergence on compact subsets, $|h'(z_0)| = \eta$. Since $\Sigma \neq \emptyset$, $\eta > 0$, h is not constant. Since $|f_n(z)| < 1$ for any $z \in U$, we get $|h(z)| \leq 1$ for any z. But by the open mapping theorem, $|h(z)| < 1$, and $h : U \to \mathbb{D}$.
It remains to prove the bijectivity. Let $z_0 \in U$, $\omega_0 := h(z_0)$, and $\omega_n := f_n(z_0)$. Let $z_1 \in U$, and $D(z_1, \epsilon) \subset U$ such that $z_0 \notin \overline{D(z_1, \epsilon)}$ and $h - \alpha$ has no zero on $\partial D(z_1, \epsilon)$ (it is possible since h is not constant, and thus no limit point). Thus $(f_n - \alpha_n)_n$ converges uniformly on $\overline{D(z_1, \epsilon)}$ to $h - \alpha$. $(f_n)_n$ are one-to-one, thus $(f_n - \alpha_n)_n$ do not have any zeros on $D(z_1, \epsilon)$. By Rouché theorem:

$$|(f_n - \alpha_n) - (h - \alpha)| < |f_n - \alpha_n|$$

for n large, thus $h - \alpha$ has no zeros on $D(z_1, \epsilon)$, and $h(z_1) \neq \alpha = h(z_0)$. Thus h is injective, and $h \in \Sigma$. $\square$

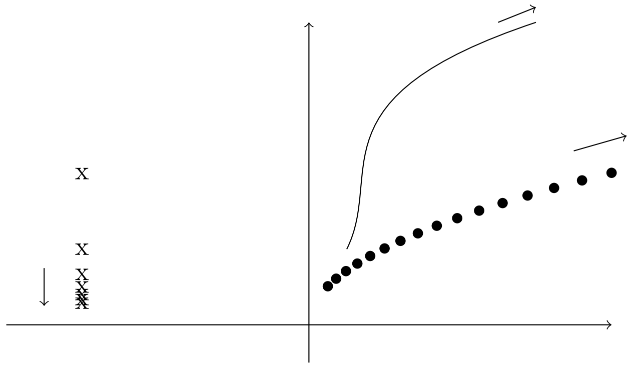
Day 15

6.6 Behaviour of the boundary [1]

We say that a sequence $(z_n)_n \in U^{\mathbb{N}}$ or an arc $z(t)$, for $0 \leq t < 1$ tends to the boundary if for any compact set $K \subset U$:

$$\begin{cases} \exists N > 0, \forall n \geq G, \quad z_n \notin K, \\ \exists a > 0, \forall t \geq a, \quad z(t) \notin K. \end{cases}$$

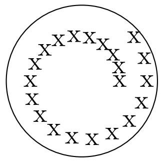
For example, on the upper half-plane $\mathbb{H}$, the sequence can tend to a point on the real axis or to infinity:



Theorem 106. *Let U_1 and U_2 be two conformally equivalent sets, with $\phi(U_1) = U_2$. If $(z_n)_n$ or $z(t)$ tends to the boundary of U_1, then $(\phi(z_n))_n$ or $\phi \circ z(t)$ tends to the boundary of U_2.*

Proof. If $(z_n)_n$ tends to the boundary of U_1, consider a compact subset K of U_2. $\phi^{-1}(K)$ is a compact set of U_1. There exists $N > 0$, such that for any $n \geq N$, $z_n \notin K$; so $\phi(z_n) \notin \phi(\phi^{-1}(K)) = K$. Thus $(f(z_n))_n$ tends to the boundary of U_2. □

This definition takes into account a large set of phenomena, including the case that there is not limit to this sequence, and the sequence goes closer to the boundary. It thus goes out of any compact set:



7 Analytic continuation

We have proved that if U and V are two regions, $U \cap V$ is not discrete, $f \in H(U)$, $g \in H(V)$, and $f = g$ on a non-discrete subset of $U \cap V$, then $f = g$ on $U \cap V$.
in this case we say ([11]) that g is a (direct) analytic continuation of f, and we also that (g, V) is a (direct) analytic continuation of (f, U).

For example, we have proved the existence of an analytic continuation of a function defined by a serie on a disk in the mandatory homework.

The assumption of a non-discrete subset is superfluous in case of two open subsets. However, the analytic continuation also holds for other types of regular subsets. For example, the in the Schwarz reflection lemma, the two sets U^+ and U^- are, and the real interval I is on the boundary of the two open sets. We proved the existence of an analytic continuation of a holomorphic on $U^+ \cup I$ to U^-, under condition on the boundary.

This analytic continuation is a useful tool when dealing with holomorphic functions with certain properties. We will deal later in more detail the case of algebraic functions, but let us give now the example of solution of linear differential equation. Consider the equation:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}'(x) = \begin{pmatrix} 0 & \frac{1}{x} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

On $\mathbb{R}$, the unique solution is defined on the maximal interval $\mathbb{R}_+^*$ by $\begin{pmatrix} \ln(x) \\ 1 \end{pmatrix}$. One natural question is thus : can we continue, on the real axis, the function before 0? Considering this problem on $\mathbb{C}$, one solution is $\begin{pmatrix} \mathrm{Log}(z) \\ 1 \end{pmatrix}$ defined on $\mathbb{C} \backslash \mathbb{R}_-$, where Log is the principal value of the logarithm. Since 0 is a (certain) singularity, the value on the negative real axis changes if we consider the analytic continuation by the upper half-plane or by the lower half-plane.(for more about Linear differential equations, see [7])

7.1 Continuation along a curve

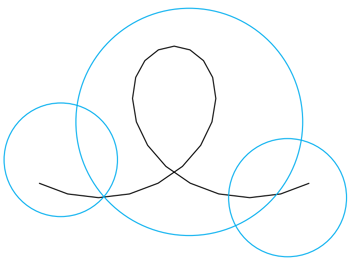
Consider an open circular disk $D_0 = D(z_0, r_0)$ and $f \in H(D_0)$. Let $\gamma : [a; b] \to \mathbb{C}$ be a path. We define the analytic continuation of (f_0, D_0) along γ by a finite sequence $((f_k, D_k))_{0 \le k \le n}$, with:

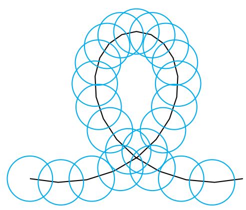
- $D_k = D(z_k, r_k)$ is an open disk, and $f_k \in H(D_k)$;
- $(D_0, \cdots, D_n)$ covers γ in the following sense:

$$\exists a = a_0, \le a_1 \le \cdots \le a_{n+1} = b, \quad \forall 0 \le k \le n, \quad \gamma\left([a_k; a_{k+1}]\right) \subset D_k.$$

- for any $k \in [\![0; n-1]\!]$, we have the equality:

$$f_{k|D_k \cap D_{k+1}} = f_{k+1|D_k \cap D_{k+1}}.$$



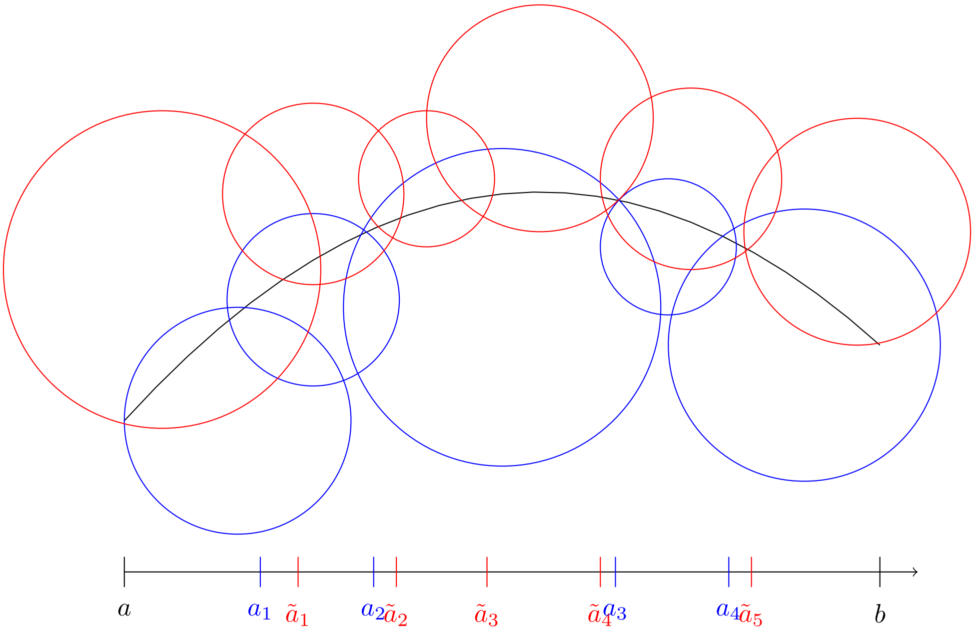


The two pictures are examples of domains for which the analytic continuation can hold. In particular, notice that two non-adjacent circles can overlap. The domain obtained by the union of circles may not be connected. This remark is useful when we want to go around one singularity.

The definition still holds when dealing with convex instead of circles.

Theorem 107. *Let $\gamma : [a, b] \to \mathbb{C}$ be a path, $((f_k, D_k))_{0 \le k \le n}$ be an analytic continuation of (f_0, D_0) along γ, and $((g_j, E_j))_{0 \le j \le m}$ be an analytic continuation of (g_0, E_0) alogn γ. If $f_0 = g_0$ on a neighbourhood of $z_0 = \gamma(a)$, then $f_n = g_m$ on $E_m \cap D_n \ni \gamma(b)$.*

Proof. The proof is similar to the extension of the local Cauchy theorem to a simply-connected region, with the covering by disks.



To $((f_k, D_k))_{0\le k\le n}$ (respectively $((g_j, E_j))_{0\le j\le m}$), we associate the decomposition $a = a_0 \le a_1 \le \cdots \le a_n = b$ (respectively $a = \tilde{a}_0 \le \tilde{a}_1 \le \cdots \le \tilde{a}_m = b$.
Define the subsequence $a = c_0 \le c_1 \le \cdots \le c_l = b$, where for any i, $c_i = a_k$ or $\tilde{a}_j$ for some k or j.
Consider $\gamma([c_0; c_1])$, we have $f_0 = g_0$ on $D_0 \cap E_0$, since $f_0 = g_0$ on a neighbourhood of $\gamma(c_0)$. Suppose $c_1 = a_1$. We have, on $\gamma([c_1, c_2])$, $f_1 = g_0$ on $D_1 \cap E_0$, since $c_1 \le \tilde{a}_1$, $\gamma(c_1) \in E_0$, thus there exists an open set $V \ni \gamma(c_1)$ with $V \subset D_1 \cap D_0 \cap E_0$.

- $g_0 = f_0$ on $D_0 \cap E_0$,
- $f_0 = f_1$ on $D_0 \cap D_1$.

Thus $g_0 = f_1$ on V. By uniqueness, $g_1 = f_0$ on $D_0 \cap E_1$. By induction, $g_m = f_n$ on $D_n \cap E_m$. □

As a consequence, we define f_γ a function defined on a neighbourhood V of $\gamma(b)$, with $f_n = f$ on $V \cap D_n$.

Consider the following disks and analytic functions:

$$D_0 = D(1,1), \quad f_0(z) = -\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}(z-1)^n,$$

$$D_1 = D(i,1), \quad f_1(z) = -\sum_{n=1}^{+\infty} \frac{i^n}{n}(z-i)^n + i\frac{\pi}{2}$$

$$D_2 = D(-1,1), \quad f_2(z) = -\sum_{n=1}^{+\infty} \frac{1}{n}(z+1)^n + i\pi$$

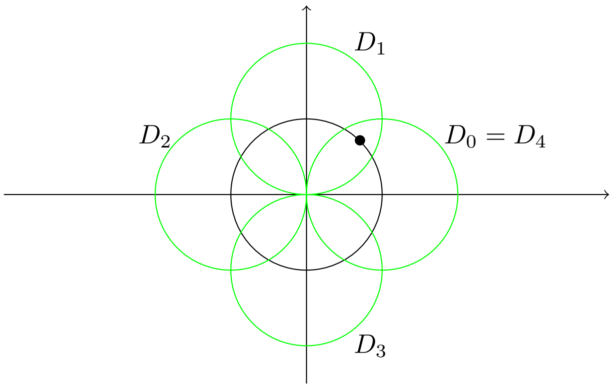
$$D_3 = D(-i,1), \quad f_3(z) = -\sum_{n=1}^{+\infty} \frac{(-i)^n}{n}(z+i)^n + i\frac{3\pi}{2},$$

$$D_4 = D(1,1), \quad f_4(z) = -\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}(z-1)^n + 2i\pi.$$

We can compose those functions with the exponential :

$$\begin{aligned}
&\exp(f_0(z)) = z;\\
&\exp(f_1(z)) = e^{i\frac{\pi}{2}} \exp\left(-\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{z}{i}-1\right)^n\right) = i\frac{z}{i} = z,\\
&\exp(f_2(z)) = e^{i\pi} \exp\left(-\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{z}{-1}-1\right)^n\right) = -1\frac{z}{-1} = z.
\end{aligned}$$

Now, define the path $\gamma : \left[0; \frac{9}{4}\pi\right] \to \mathbb{C}$, with $\gamma(t) = e^{it}$. Notice that, for example, $\gamma(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \in D_0 \cap D_1$.



Thus $((D_k, f_k))_k$ is an analytic continuation of (D_0, f_0) along γ.

Remark 108. *An analytic continuation may not be a function!*

$$f_0(1) = 0 = \ln(1) + i0, \quad f_4(1) = 2i\pi + f_0(1) = 2i\pi.$$

In fact, $f_0 = Log_{|D_0}$*, and* $f_\gamma(t) = 2i\pi + Log_{|D_0}(t)$ *on* $D_4 = D_0$*.*

(for more about analytic continuation on Riemann surfaces, see [14])
You can for example draw the picture of $\gamma(t) \in \mathbb{C} \mapsto \ \mathrm{Im}(f_k(\gamma(t)) \in \mathbb{R}$.

Day 16

We continue with the application to algebraic functions.
We consider a polynomial P in two variables $P(T_1, T_2)$ with $P \neq 0$. An algebraic function is a function f defined on an open set U such that there exists P, such that for any $z \in U$, we have $P(f(z), z) = 0$.

Consider the function $z \mapsto \sqrt{z} = \exp(\frac{1}{2}\mathrm{Log}(z))$ on $D(1,1)$. This function is algebraic, with $P(T_1, T_2) = T_1^2 - T_2$.

Theorem 109 ([11])**.** *Let* $P(T_1, T_2)$ *be a polynomial, and* γ *be a curve. Let* $f \in H(U)$*, with* $\gamma(a) \in U$*. Suppose that* f *has an analytic continuation along* γ*, and* f_γ *is defined on a neighbourhood* V *of* $\gamma(b)$*. If* $P(f(z), z) = 0$ *on* U*, then* $P(f_\gamma(z), z) = 0$ *on* V*.*

Proof. If $((f_k, D_k))_{0 \le k \le n}$ is an analytic continuation of (f_0, D_0), then $((z \mapsto P(f_k(z), z), D_k))_{0 \le k \le n}$ is an analytic continuation of $(P(f_0, \cdot), D_0)$. An particular, since $P(f_0(z), z) = 0$ on U, then by uniqueness, $P(f_k(\cdot), \cdot) = 0$ on D_k. $\square$

Theorem 110 (Monodromy theorem)**.** *Let* U *be a region,* f *analytic on a neighbourhood* V *of* $z_0 \in U$*. Let* γ, η *be two paths with beginning point* z_0 *and end point* z_1*. If* γ *and* η *are homotopic, and* f *admits an analytic continuation along any path in* U*, then* f_γ *and* f_η *are equal in some neighbourhood of* z_1*.*

We already have an example of a function for which two homotopic paths will not give the same functions on the end of the paths. On the annulus $\mathcal{A}_{\frac{1}{2},5}$, and the paths $\gamma : t \to e^{it}$, $\eta : t \mapsto 2 + e^{i\pi + it}$ on $[0, 2\pi]$, if $f = Log$ on a neighbourhood of 1, then:

$$f_\eta = \mathrm{Log}, \quad \text{and} \quad f_\gamma = 2\pi i + \mathrm{Log}$$

on a neighbourhood of 1.

Proof. Let the homotopy $\psi : [a,b] \times [c,d] \to U$, with:

$$\begin{cases} \psi(\cdot, c) = \gamma, \\ \psi(\cdot, d) = \eta. \end{cases} \tag{9}$$

Consider the set $S := \left\{u \in [c;d]; f_{\psi(\cdot,c)} = f_{\psi(\cdot,u)} \text{ on a small disk around } z_1\right\}$. You see me coming, we will use a connectedness argument.

- $S \neq \emptyset$ since $c \in S$.

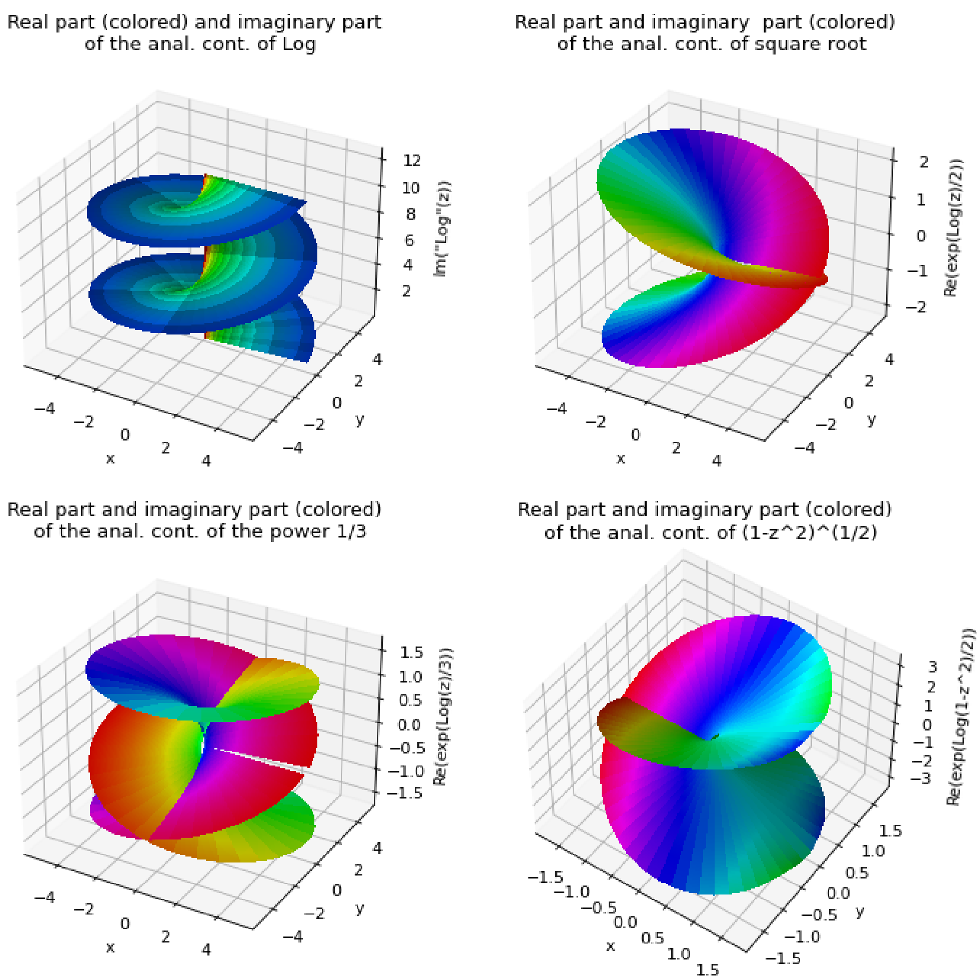
- S is open in $[c;d]$. Let $u \in S$, and $\gamma_u := \psi(u)$. Let $(D_0, \cdots, D_n)$ be a covering of γ_u associated with the analytic continuation of f along γ_u. For u' close enough to u, this covering is also a covering of $\gamma_{u'}$, and the analytic continuation $((f_k, D_k))_k$ is also on an analytic continuation for any u' close enough to u (because ψ is continuous).

- S is closed. If u is in the closure of S, by the same argument, any point "close enough" to u satisfies $f_{\psi(\cdot,u)} = f_{\psi(\cdot,u')}$ (on a small ball around z_1). Thus $u \in S$.

By connectedness, $S = [c, d]$. □

Corollary 111. *Let U be a simply-connected region, $z_0 \in U$, f analytic in a neighbourhood of z_0. Suppose that f can be analytically continued along any path γ with beginning point z_0. For any $z \in U$, define γ_z a path from z_0 to z. Then the function $z \mapsto f_{\gamma_z}(z)$ is in $H(U)$.*

Proof. This definition is independent of γ_z, since if γ_z and $\tilde{\gamma}_z$ have the same beginning and end points, they are homotopic. By the monodromy theorem, the function f_γ is independent of the path γ.
The function $g : z \mapsto f_{\gamma_z}(z)$ is thus well-defined. Is this function analytic?
Let D_n be such that $z_1 \in D_n \subset U$, and γ a path from z_0 to z_1. We prove that on D_n, $g(z) = f_{\gamma_{z_1}}(z)$, for any $z \in D_n$, and the function $f_{\gamma_{z_1}}$ is analytic on D_n. Consider a point $z \in D_n$. Let η be a path from z to z_1, and $\tilde{\gamma} = \gamma \cup \bar{\gamma}$, where $\bar{\gamma}$ is an interval from z_1 to z; thus $\bar{\gamma}^* \subset D_n$. $\tilde{\gamma}$ and η are homotopic, thus $f_\eta = f_{\tilde{\gamma}}$ around z, and the function can be analytically continued on D_n. Finally, $g(z) = f_{\gamma_{z_1}}(z)$ on D_n. □



7.2 Premises on elliptic functions ([18], see also [1])

This section initiates the little Picard theorem. In this part, we build a function on a fundamental domain Q, and then extend it to the upper half-plane. The construction of this function λ involves the Schwarz reflection lemma, and the proof of the little Picard theorem also involves analytic continuation, based on the function λ.

To define the domain, we use a subgroup of the fundamental group. The modular group G is the following set of linear fractional transformations:

$$G := \left\{ \phi \in Aut(\hat{\mathbb{C}}); \quad \phi(z) = \frac{az+b}{cz+d}, \quad (a,b,c,d) \in \mathbb{Z}^4, \quad ad-bc=1 \right\}.$$

Since the coefficients are real, we have $\psi(\mathbb{R} \cup \infty) = \mathbb{R} \cup \infty$. Let $\mathbb{H}^+$ be the upper half-plane; we have:

$$\phi(i) = \frac{ai+b}{ci+d} = \frac{(a+ib)(-ci+d)}{c^2+d^2} = \frac{-ac+bd+i(ad-bc)}{c^2+d^2},$$

$\mathrm{Im}(\phi(i)) > 0$, thus $\phi(\mathbb{H}^+) = \mathbb{H}^+$.
Notice that the two applications:

$$T : z \mapsto z+1, \quad \text{and} \quad S : z \mapsto -\frac{1}{z}$$

generate the group G.

Remark 112. *Can we define a fundamental domain of G? To do so, we need a clear definition of a fundamental domain (see later for the definition for another subgroup). I feel like the different references avoid the problem : [1] is not clear, [18] hide it, [11] does not talk about it. The problem is the non-uniqueness of the images of some points : we have $S(i) = i$, and $S(e^{-i\frac{\pi}{3}}) = T(e^{-i\frac{\pi}{3}})$. Thus, we can not ask of a fundamental neither to be open, nor to be close, nor to get the uniqueness of the orbit of one point. Maybe the only requirement on the uniqueness holds on the interior of Q.*

Consider the following subgroup $\Gamma \subset G$, generated by:

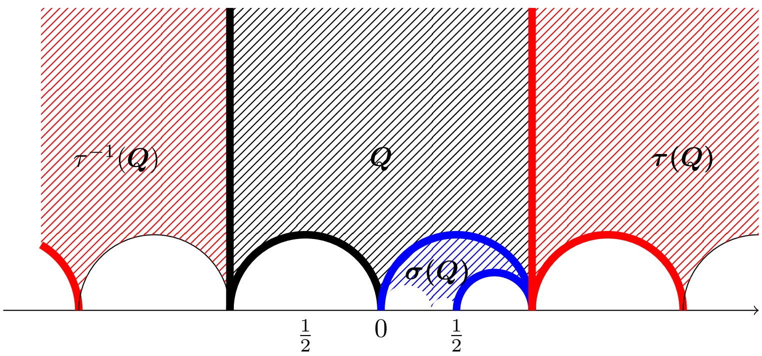
$$\sigma(z) = \frac{z}{2z+1}, \quad \tau(z) = z+2.$$

(in the chapter on elliptic functions, we will use the matrices notations : for the function σ, the associated matrices are $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$, and for τ : $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix}$).
We can compute the inverse of the Möbius transformation σ:

$$\sigma^{-1}(z) = \frac{z}{-2z+1}.$$

Let us define the domain:

$$Q := \left\{ z \in \mathbb{H}^+; y > 0, \quad -1 \leq x < 1, \quad \left|z + \frac{1}{2}\right| \geq \frac{1}{2}, \quad \left|z - \frac{1}{2}\right| > \frac{1}{2} \right\}.$$



We can compute the image of this domain by the different applications. If you consider τ or τ^{-1}, the domain Q is translated of 2 or -2. For the function σ, consider the image of the boundary. Let $\mathcal{C} = \left\{ z \in \mathbb{H}^+; |z + \frac{1}{2}| = \frac{1}{2} \right\}$. We have:

$$\sigma(0) = 0, \quad \sigma(-1) = 1, \quad \sigma(-\frac{1}{2} + \frac{i}{2}) = \frac{1}{2} + \frac{i}{2},$$

and a circle (defined by three points) is uniquely send on a circle of $\hat{\mathbb{C}}$. Let consider now the real half-line $\mathcal{D} = \{z \in \mathbb{H}^+; z = -1 + iy, \quad y \geq 0\}$. We have:

$$\sigma(-1) = 1, \quad \sigma(\infty) = \frac{1}{2}.$$

By using the conservation of the angle between $\mathcal{C}$ and $\mathcal{D}$, which is 0, we have $\sigma(\mathcal{D}) = \left\{z \in \mathbb{H}^+; |z - \frac{3}{4}| = \frac{1}{4}\right\}$. Similarly, we have $\sigma(1) = \frac{1}{3}$, and we can identify the boundary of $\sigma(Q)$.

Q is called a fundamental domain of Γ, because it satisfies the two first conditions of the following theorem:

Theorem 113 ([18]). *1. If ϕ_1 and ϕ_2 are distinct applications of Γ, then $\phi_1(Q) \cap \phi_2(Q) = \emptyset$.*

2. $\bigcup_{\phi \in \Gamma} \phi(Q) = \mathbb{H}^+$.

3. Γ contains all the applications $\phi \in G$ of the form:

$$\phi(z) = \frac{az + b}{cz + d}, \tag{10}$$

for which a and d are odd integers, b and c are even.

Proof. Let Γ_1 be the subset of G satisfying (10). We can verify (do it) that Γ_1 is a subgroup. Furthemore, since σ and τ are in Γ_1, it implies that $\Gamma \subset \Gamma_1$.
Define now by 1') the statement 1) with the group Γ_1. If:

1. ' $\phi_1 \neq \phi_2 \in \Gamma_1$, then $\phi_1(Q) \cap \phi(Q) = \emptyset$.
2. $\bigcup_{\phi \in \Gamma} \phi(Q) = \mathbb{H}^+$

then Γ is not a proper subgroup of Γ_1. It thus concludes the theorem, and it remains to prove 1') and 2).
Proof of 1'): If ϕ_1, ϕ_2 are in Γ_1, with $\phi_1 \neq \phi_2$, then $\phi := \phi_1^{-1} \circ \phi_2 \in \Gamma_1$, $\phi \neq id$, and $\phi_1(Q) \cap \phi_2(Q) = \emptyset$ if and only if $Q \cap \phi(Q) = \emptyset$.
It is thus sufficient to prove 1') with ϕ and id, $\phi(z) = \frac{az+b}{cz+d}$.
First case: $c = 0$. $\phi(z) = \frac{a}{d} z + \frac{b}{d}$, $ad = 1$, thus $\phi(z) = z + 2n$ for some $n \in \mathbb{Z}^*$. In this case, $\phi(Q) \cap Q = \emptyset$, see the previous remark.
Second case: $c = 2d$, with $c \wedge d = 1$, $c = \pm 1$ and $d = \pm 1$. Thus:

$$\phi(z) = \pm \frac{az + b}{2z + 1} = \frac{(1 \pm 2b)z \pm b}{2z + 1} = \sigma(z) + 2m,$$

for some $m \in \mathbb{Z}^*$. Since $\sigma(Q) \subset \bar{D}(\frac{1}{2}, \frac{1}{2})$, we have $\sigma(Q) + 2m \cap Q = \emptyset$.
Third case: $c \neq 0$ and $c \neq 2d$. We claim that $|cz + d| > 1$ for any $z \in Q$. Otherwise, we would have:

$$|cz + d| \leq 1 \quad \Leftrightarrow \quad |z + \frac{d}{c}| \leq \frac{1}{|c|} \quad \Leftrightarrow \quad z \in \overline{D\left(-\frac{d}{c}, \frac{1}{|c|}\right)}.$$

(we exclude the case $-\frac{d}{c} = \pm \frac{1}{2}$.) If $\overline{D\left(-\frac{d}{c}, \frac{1}{|c|}\right)} \cap Q \neq \emptyset$, with a radius $r = \frac{1}{|c|} \leq \frac{1}{2}$, then $D(-\frac{d}{c}, \frac{1}{|c|})$ contains 0, 1 or -1.
Thus one of the three possibilities hold:

- $|d| < 1$: absurd by definition of d.
- $|-c + d| < 1$ or $|c + d| < 1$: absurd since $c \in 2\mathbb{Z}$, $d \in 2\mathbb{Z} + 1$.

Thus $|cz + d| > 1$, and it implies:

$$\mathrm{Im}(\phi(z)) = \frac{1}{|cz + d|^2} \mathrm{Im}\left(bd + acz\bar{z} + zad + \bar{z}bc\right) = \frac{\mathrm{Im}(z)}{|cz + d|^2} < \mathrm{Im}(z), \quad \forall z \in Q. \tag{11}$$

Day 17

Now we prove that $\phi(Q) \cap Q = \emptyset$. If $z \in Q$ and $\phi(z) \in Q$, we study ϕ^{-1}. The first and second cases for ϕ^{-1} are not possible if ϕ satisfies the third case. Thus ϕ^{-1} also satisfies the third case, and we have:

$$\mathrm{Im}(z) = \mathrm{Im}(\phi^{-1} \circ \phi(z)) < \mathrm{Im}(\phi(z)).$$

Thus, $\mathrm{Im}(z) < \mathrm{Im}(z)$ which is a contradiction, and thus $Q \cap \phi(Q) = \emptyset$, and proves 1').

We now prove 2). Let us define:

$$\Sigma := \bigcup_{\phi \in \Gamma} \phi(Q).$$

Since $\tau \in \Gamma$, $\tau^n \in \Gamma$, thus:

$$\left\{ z \in \mathbb{H}^+; |2z - (2m+1)| > 1, \forall m \in \mathbb{Z} \right\} \subset \Gamma.$$

In fact, the previous inclusion also holds with $\geq$ instead of $>$, since $\sigma\left(\mathcal{C}\left(-\frac{1}{2}, \frac{1}{2}\right)\right) = \mathcal{C}\left(-\frac{1}{2}, \frac{1}{2}\right)$.
Let $\omega \in \mathbb{H}^+$. We prove that $\omega \in \Sigma$, by finding an application which sends ω to a point in its orbit with the highest imaginary part (thus in a $\tau^n(Q)$ for some n).
Since $\mathrm{Im}(z) > 0$, there are only finitely many pairs $(c, d) \in \mathbb{Z}^2$ such that $|c\omega + d|$ lies below a certain bound. Choose ϕ_0 such that $|c\omega + d|$ is minimized:

$$\forall \phi \in \Gamma, \quad \mathrm{Im}\left(\phi(\omega)\right) = \frac{\mathrm{Im}(\omega)}{|c_\phi \omega + d_\phi|^2} \leq \mathrm{Im}(\phi_0(z)).$$

Let $z_0 := \phi_0(\omega)$ (this is a point with highest imaginary part). We thus have:

$$\begin{aligned} \forall \phi \in \Gamma, &\quad \mathrm{Im}(\phi \circ \phi_0(z_0)) \leq \mathrm{Im}(z_0), \\ \forall \phi \in \Gamma, &\quad \mathrm{Im}(\phi(z_0)) \leq \mathrm{Im}(z_0). \end{aligned}$$

With $\phi := \sigma \circ \tau^{-n}$ and $\phi := \sigma^{-1} \circ \tau^{-n}$:

$$\left(\sigma \circ \tau^{-n}\right)(z_0) = \frac{z_0 - 2n}{2z_0 - 4n + 1}, \quad \left(\sigma^{-1} \circ \tau^{-n}\right)(z_0) = \frac{z_0 - 2n}{-2z_0 + 4n + 1},$$

and by the previous computation on the imaginary part:

$$\mathrm{Im}(\sigma \circ \tau^{-n})(z_0) = \frac{\mathrm{Im}(z_0)}{|2z_0 - 4n + 1|^2} \leq \mathrm{Im}(z_0),$$

we obtain:

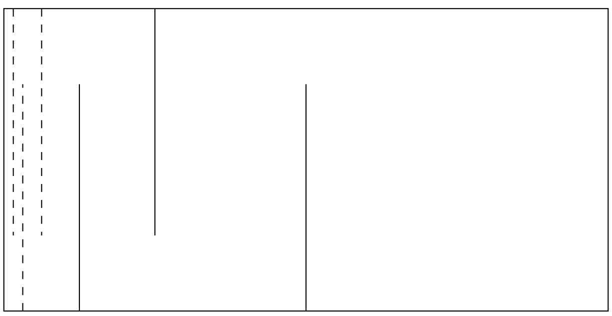
$$\forall n \in \mathbb{Z}, \quad |2z_0 - 4n + 1| \geq 1, \quad |-2z_0 + 4n + 1| \geq 1.$$

Thus z_0 is in Γ, so is ω. This concludes the proof. □

Next, we define an application on Q. To do so, we need to know the behaviour at the boundary of Q.

Definition 114. *Let U be a simply-connected region, and $\beta \in \partial U$ a point in the boundary. β is a simple boundary point, if for any sequence $(z_n)_n \in U^{\mathbb{N}}$, $z_n \to \beta$, there exists a curve $\gamma : [a, b] \to \mathbb{C}$, $\gamma(b) = \beta$, $\gamma([a, b]) \subset U$, and a sequence $a \leq a_0 < a_1 < \cdots < a_n < \cdots < b$, such that for any n, $\gamma(a_n) = z_n$.*

In the following example, U is simply-connected, but any point in $[0, i]$ is not simple:



We will need the following theorem:

Theorem 115 ([18], see also [13])**.** *Let ϕ be a conformal application from a simply connected set U onto $\mathbb{D}$, with only simple boundary points. Then ϕ extends to a homeomorphism onto $\bar{\mathbb{D}}$.*

Since the proof of this theorem is tedious, we only explain some steps. The first step focuses on the continuity at only one simple boundary point:

Theorem 116 ([18] in exercise)**.** *If U is a simply connected region, ϕ a conformal mapping of U onto $\mathbb{D}$. If $\beta \in \partial U$ is a simple boundary point, then ϕ extends continuously to $U \cup \{\beta\}$. Furthermore, $|\phi(\beta)| = 1$.*

The second step states that two boundary points can not have the same image on the unit circle by ϕ:

Corollary 117 ([18, 13])**.** *If $\beta_1 \in \partial U$, $\beta_2 \in \partial U$, $\beta_1 \neq \beta_2$ and β_1 and β_2 are simple boundary points, then $\phi(\beta_1) \neq \phi(\beta_2)$.*

The proof is based on the Lindelöf theorem.
Finally, the third step considers the continuity of the application.

It is now time to define the modular function λ:

Theorem 118. *Let Γ and Q be as before. There exists a function $\lambda \in H(\mathbb{H}^+)$ such that:*

- *$\forall \phi \in \Gamma$, $\lambda \circ \phi = \lambda$.*
- *λ is one-to-one on Q.*
- *$Im(\lambda)$ ($= \lambda(Q)$ by its definition) is equal to $\mathbb{C} \backslash \{0, 1\}$.*

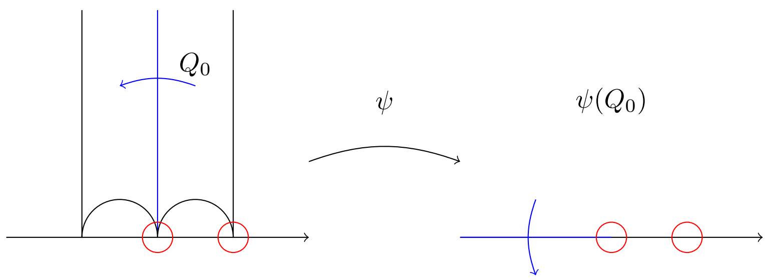
Proof. Let us define the following set:

$$Q_0 := \left\{ z \in \mathbb{H}; \left| z - \frac{1}{2} \right| > \frac{1}{2};\ \mathrm{Re}(z) \in (0,1) \right\} \subset Q.$$

Let $\phi_1 \in Aut(\hat{\mathbb{C}})$ such that $\phi_1(\infty) = 1$, $\phi_1(-i) = \infty$. Then $\phi_1(Q_0)$ is a simply connected region of $\mathbb{C}$, with simple boundary points. It is furthermore bounded. Secondly, there exists ϕ_2 a conformal mapping from $\phi_1(Q_0)$ onto $\mathbb{D}$. There also exists ϕ_3 a conformal mapping from $\mathbb{D}$ onto $\mathbb{H}^+$ (we can also choice the images of three chosen points of the unit circle.
Let $\psi := \phi_3 \circ \phi_2 \circ \phi_1 : Q_0 \to \mathbb{H}^+$; it is a conformal mapping, which can be continuously extended at the boundary by the previous continuation, and we can choose:

$$\psi(0) = 0, \quad \psi(1) = 1, \quad \psi(\infty) = \infty.$$

Thus, $\phi\left(\{z \in \mathbb{H}^+; \mathrm{Re}(z) = 0\}\right) = (-\infty; 0) \subset \mathbb{R} \subset \mathbb{C}$.



Day 18

By the Schwarz reflection principle, ψ cann be extended to the interior of Q by $\psi(-x+iy) = \overline{\psi(x+iy)}$. By this definition, we have the extension of ψ to Q, with:

$$\psi\left(\{z \in \mathbb{H}^+; \mathrm{Re}(z) = -1\}\right) = (1, +\infty) \subset \mathbb{R}, \tag{12}$$

$$\psi\left(\left\{z \in \mathbb{H}^+; |z + \frac{1}{2}| = \frac{1}{2}\right\}\right) = (0, 1) \subset \mathbb{R}. \tag{13}$$

Also :

$$\psi(-1+iy) = \psi(1+iy) = \psi(\tau(-1+iy)), \quad \text{for any } y \in \mathbb{R}_+^*;$$

$$\psi(-\frac{1}{2} + \frac{e^{i\theta}}{2}) = \psi\left(\frac{1}{2} + e^{i(\pi-\theta)}\right) = \psi\left(\sigma\left(-\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)\right), \text{ for any } \theta \in (0, \pi),$$

by the limits and the definition of ϕ, σ and θ.
Let us now define:

$$\lambda(z) := \psi \circ \phi^{-1}(z), \quad \forall z \in \phi(Q), \forall \phi \in \Gamma.$$

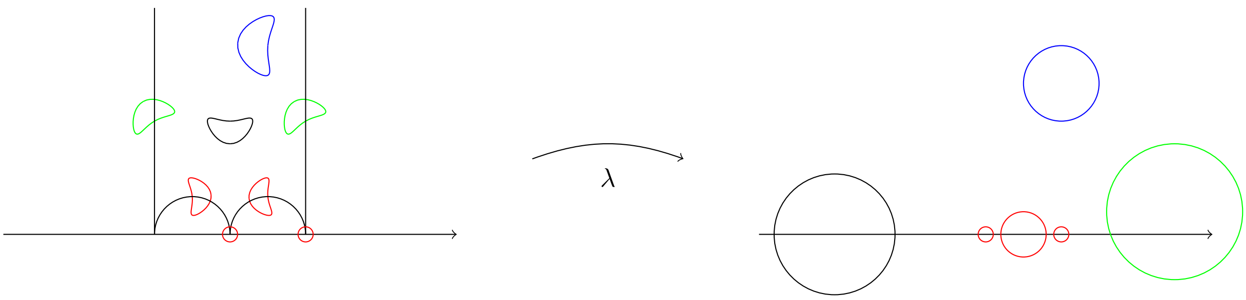
This function is well-defined, since each $z \in \mathbb{H}^+$ has a unique $\phi \in \Gamma$ such that $z \in \phi(Q)$. λ satisfies the first and third points of the theorem, and λ is holomorphic on the interior of $\phi(Q)$. By (12) and (13), $\lambda \in \mathcal{C}^0\left(Q \cup \sigma^{-1}(Q) \cup \tau^{-1}(Q)\right)$. The only point to check is the holomorphicity of λ. In fact, by the construction, we have used the Schwarz reflection lemma along the imaginary axis, which involves a reflection of the image set along $(-\infty, 0)$. The same procedure can be applied along the axis $(1 + iy; y \in \mathbb{R}^+)$, and the reflection of the image set along the axis $(1; +\infty)$. Notice that by the definition of the reflection, the values are coherent with $\tau(Q)$, and thus prove the holomorphic property on this axis. Same along the half-circle centered at $\frac{1}{2}$ and of radius $\frac{1}{2}$, which corresponds to a symmetry along $(0, 1)$, and proves the holomorphy on this half-circle. We then conclude by the group action of Γ. □

(the last step of [18] uses the Morera theorem instead of the reflection along the different axis; this is not clear, since this theorem applies to open sets, and we cannot separate a path along the boundary).

7.3 The Little Picard theorem

Theorem 119 ([18])**.** *If $f \in H(\mathbb{C})$ (entire function), and if two points in $\mathbb{C}$ are not in the range of $\mathbb{C}$, then f is a constant function.*

Proof. If $a, b \notin \mathrm{Im}(f)$, we can replace f by $\frac{f(z)-a}{b-a}$, so that (a, b) is replaced by $(0, 1)$. f maps $\mathbb{C}$ on $U := \mathbb{C}_{\backslash\{0,1\}}$.
Consider a disk $D_1 \subset U$. There exists a region $V_1 \subset \mathbb{H}^+$, such that $\lambda(V_1) = D_1$, and λ is one-to-one from V_1 to D_1.



Each V_1 intersects at most two of the domains $\{\phi(Q), \phi \in \Gamma\}$.
To each V_1, there exists a unique $\psi_1 : D_1 \to V_1$ which is holomorphic (conformal), with $\lambda(\psi_1(z)) = z$ for any $z \in D_1$. Let D_2 be another disk in U, with $D_1 \cap D_2 \neq \emptyset$. We can choose similarly V_2, with $V_2 \neq V_1 \neq \emptyset$. Thus (ψ_2, D_2) is a direct analytic continuation of (ψ_1, D_1).
There exists A_0, disk centered at 0, such that $f(A_0)$ is in a disk $D_0 \subset U$. Choose ψ_0 as above; $\lambda(\psi_0(z)) = z$ for any $z \in D_0$. Let $g(z) := \psi_0(f(z))$ on A_0.
Let γ be a curve which begins at 0; $f \circ \gamma$ is a curve in U. $f \circ \gamma$ can be covered by a finite sequence of disks, and by the previous method, g can be analytically continued along any path in $\mathbb{C}$.
Since the plane is simply connected, the monodromy theorem implies that g extends to an entire function, and $\mathrm{Im}(g) \subset \mathbb{H}^+$. Also $\Psi : z \mapsto \frac{z-i}{z+i}$ conformally maps $\mathbb{H}^+$ onto $\mathbb{D}$, so $\Psi \circ g : \mathbb{C} \to \mathbb{D}$ is holomorphic and bounded.
By Liouville theorem, it is constant, thus g is constant. ψ_0 was bijective from D_0, thus f is constant on A_0, which is a non-discrete set, and thus f is constant.

□

Corollary 120 ([4])**.** *If $f \mapsto \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a meromorphic function, and $\hat{\mathbb{C}}_{\backslash f(\mathbb{C})}$ has at least three points, then f is constant.*

Proof. Suppose that f misses the points $(a, b, c) \in \hat{\mathbb{C}}^3$. There exists a Möbius transformation ψ satisfying:

$$\psi(a) = 0, \quad \psi(b) = 1, \quad \psi(c) = \infty.$$

$\psi^{-1} \circ f \in H(\mathbb{C})$, and misses at least two points. By the previous theorem, it is constant. □

For example, consider the function the meromorphic function $f(z) = (\sin(z))^{-1}$. It has an infinite number of poles. It thus misses at most three values, including 0.

8 The Dirichlet problem

We have been dealing a lot for now with holomorphic functions, but we let aside the mean value property. In fact, functions which do not satisfy the mean value property, but with an inequality instead of an equality, have their own interest to solve the Dirichlet problem.
We have studied the case of the disk, with regular enough boundary conditions. However, when the boundary is a more eclectic, we do not have enough tools yet to solve the problem; for example, what happens for the case of the punctured disk $B(0,1)\backslash\{0\}$?

8.1 Harmonic conjugate [4]

Definition 121. *Let U be a region of $\mathbb{C}$, $f \in H(U)$, with $f = u + iv$. u and v are called harmonic conjugate. We denote the set of harmonic functions on U by $Har(U)$.*

Proposition 122. *If $h : U \to U'$ is conformal, and $v \in Har(U')$, then $v \circ h \in Har(U)$.*

We have already the proof when dealing with the Riemann mapping theorem.

Theorem 123. *Let U be a simply-conneccted region, $v \in Har(U)$. v possesses a harmonic conjugate u (there exists $f \in H(U)$ with $f = u + iv$).*

Proof. **First case:** $U = \mathbb{D}$. If $f = u + iv$ satisfies the Cauchy-Riemann equation $\partial_x u = \partial_y v$, we define u by:

$$u(x,y) := \int_0^x \partial_y v(x', y)dx' + \phi(y).$$

This function is well-defined on $\mathbb{D}$ (this is where we use the assumption on the domain). We investigate a necessary condition on ϕ:

$$\begin{aligned}\delta v = 0 \Leftrightarrow \partial_y u(x) &= \int_0^x \partial_y^2 v + \partial_y \phi \\ \Leftrightarrow \partial_y u(x) &= -\int_0^x \partial_x^2 v + \partial_y \phi \\ &\Leftrightarrow -\partial_x v(x) + \partial_y \phi + \partial_x v(0,y).\end{aligned}$$

We thus define ϕ by $\phi(y) := -\int_0^y \partial_x v(0,y')dy'$ (Here, we also used the assumption of being on a disk). Thus, the function:

$$u(x,y) := \int_0^x \partial_y v(x,y')dx' - \int_0^y \partial_x v(0,y')dy'$$

is a harmonic conjugate.
Second case: Consider any U. By the Riemann mapping theorem, there exists a conformal mapping h such that $h(U) = \mathbb{D}$. The function $v_1 := v \circ h$ is harmonic on $\mathbb{D}$. Since $f_1 := u_1 + iv_1$ is holomorphic on $\mathbb{D}$, $f := f_1 \circ h$ is harmonic on U. ☐

It is convenient to study $f \in H(U)$ or $u, v \in Har(U)$.

8.2 Harnack's theorem[4]

Lemma 124 (Harnack's inequalities)**.** *If $u \in Har(B(a,R))$ is real valued, and continuous on the closure $\overline{B(a,R)}$, and $u \geq 0$, then:*

$$\forall r \in [0,R), \forall \theta, \quad \frac{R-r}{R+r} u(a) \leq u(a + re^{i\theta}) \leq \frac{R+r}{R-r} u(a).$$

Proof. By using the formula of the Poisson kernel on v, with $v(x,y) := u\left(\frac{(x,y)-a}{R}\right)$:

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r^2} u(a + re^{i\theta})dt$$

and we conclude with:

$$\frac{R-r}{R+r} \leq \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \leq \frac{R+r}{R-r}, \quad \text{and} \quad u(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + Re^{it})dt.$$

☐

Proposition 125. *Let U be a region. $Har(U)$ is a closed subset of $\mathcal{C}(U,\mathbb{R})$, with the topology of the convergence: $(u_n)_n$ converges to u in $\mathcal{C}(U,\mathbb{R})$ if for any compact set K in U, $(u_n)_n$ converges uniformly to u on K.*

Proof. We already know that if $(u_n)_n \subset Har(U)^{\mathbb{N}} \subset \mathcal{C}(U,\mathbb{R})^{\mathbb{N}}$ converges uniformly on any compact set to u, then $u \in \mathcal{C}(U,\mathbb{R})$. It remains to prove that the limit is harmonic, or equivalently, that u satisfies the mean value property. If we consider the (compact subset) boundary $\partial B(a,R)$, where $\overline{B(a,R)} \subset U$, we have:

$$u(a) = \lim_{n\to+\infty} u_n(a) = \lim_{n\to+\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(a+re^{it})dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+Re^{it})dt.$$

□

Theorem 126 (Harnack)**.** *If $(u_n)_n \in Har(U)^{\mathbb{N}}$, with $u_1 \leq u_2 \leq \cdots$, then:*

- *either for any compact K in U, $(u_n)_n$ converges uniformly on K to $+\infty$.*
- *or $(u_n)_n$ converges (with the previous topology) to a harmonic function.*

Remark 127. *In [4], a second point of Harnack's theorem is the completeness of $Har(U)$. Defining this notion needs another requirement on the space. For example, having a metric is sufficient to define the notion of completeness. On $\mathcal{C}(U,\mathbb{R})$, we can add the metric with an infinite number of semi-norms, based on a countable familyt of compact subsets of U. In this course, we do not emphasize those notions, but it is intersting to get the completeness on the set of harmonic functions.*

Proof. We suppose $u_1 \geq 0$ (if not, consider $u_n \leftarrow u_n - u_1$). We define $u(z) := \sup_n u_n(z)$, for any $z \in U$; either $u(z) = \infty$ or $u(z) < \infty$.
Let $A := \{z \in U; u(z) = \infty\}$, and $B := \{z \in U; u(z) < \infty\}$. We now prove that A and B are open.
Let $z_0 \in U$, and R such that $\overline{B(z_0,r)} \subset U$. The Harnack's inequalities imply:

$$\forall z \in B(z_0,R), \quad \frac{R-|z-z_0|}{R+|z-z_0|}u_n(z_0) \leq u_n(z) \leq \frac{R+|z-z_0|}{R-|z-z_0|}.$$

If $z_0 \in A$, then the left hand-side tends to $+\infty$, and $B(z_0,R) \subset A$ and A is open. If $z_0 \in B$, then the right-hand-side if finite, so $B(z_0,R) \subset B$, and B is open.
By connectedness, $U = A \cup B$, $A \cap B = \emptyset$, A and B are open, and since U is connected, then either $U = A$ or $U = B$.
Consider now that $U = A$, and $u = +\infty$. Let $M \in (0,+\infty)$. We prove that:

$$\forall K \text{ compact of } U, \exists N > 0, \forall n \geq N, \forall z \in K, \quad u_n(z) \geq M.$$

We can restrict ouselves to $\overline{B(z_0,R)} \subset U$, where $z_0 \in U$ (since any compact set can be covered a finite number of closed balls of U). Let N be such that $u_N(z_0) \geq 2M$, so for any $n \geq N$, $u_n(z_0) \geq 2M$.

$$\forall z \in \overline{B(z_0,\frac{R}{2})}, \quad \frac{R-\frac{R}{2}}{R-0}u_n(z_0) \leq u_n(z),$$

and so $u_n(z) \geq M$. It implies that $(u_n)_n$ converge uniformly on compact subsets to $+\infty$.
For the case $U = B$, for any $z \in U$, define the point-wise limit: $u(z) := \lim_n u_n(z) < \infty$. We want to prove that $(u_n)_n$ uniformly converges on any compact set K to U. By the same argument, we can focus on the case of a closed ball $\overline{B(z_0,R)} \subset U$. Let $\epsilon > 0$, and $N > 0$ such that $0 \leq -u_n(z_0) + u(z_0) < \frac{\epsilon}{3}$. We thus have:

$$\forall m \geq n \geq N, \forall \rho < \frac{R}{2}, \forall z \in B(z_0,\rho), \quad 0 \leq (u_m - u_n)(z) \leq \frac{R+\rho}{R-\rho}(u_m - u_n)(z_0) \leq \epsilon.$$

Thus, since the convergence holds on closed balls, by the Heine-Borel lemma, it also holds on any compact subsets. □

8.3 Subharmonic functions [4]

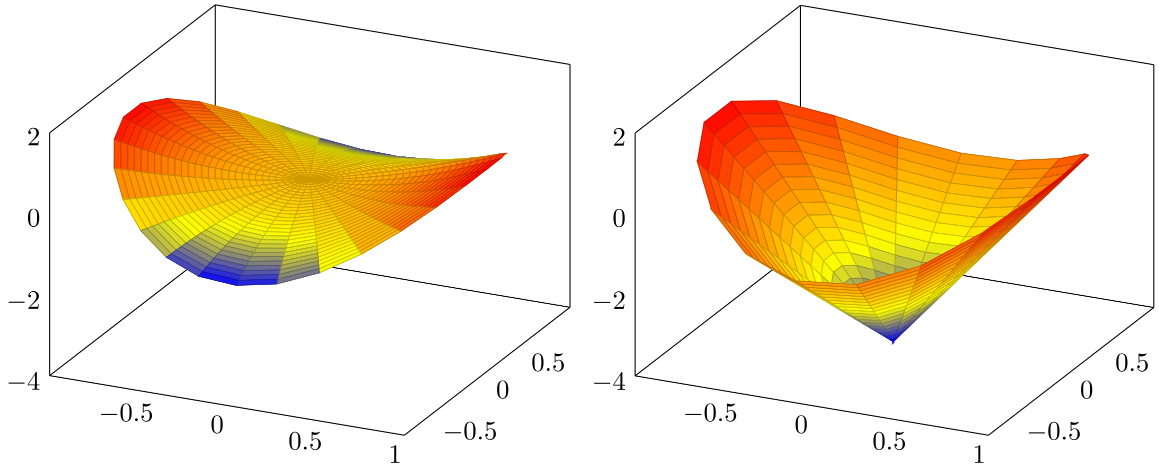
Definition 128. *Let ϕ and ψ be two continuous functions from U to $\mathbb{R}$.*
ϕ is subharmonic if:

$$\forall \overline{B(a,r)} \subset U, \quad \phi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(a+re^{i\theta})d\theta.$$

ψ is superharmonic if:

$$\forall \overline{B(a,r)} \subset U, \quad \psi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \psi(a+re^{i\theta})d\theta.$$

For example, consider the function $u(x, y) = x^2 - y^2$ on $B(0, 1)$. It is harmonic. However, if you consider the function v with $v = u$ on the unit disk, $v(0) = -4$, and if for any θ, $r \mapsto v(re^{i\theta})$ is a line, then this function is subharmonic. At each point, the value is less than the average of this function on a circle centered at this point.



Remark 129. *ϕ is subharmonic if and only if $-\phi$ is superharmonic. We will thus focus on subharmonic functions.*
If u_1 and u_2 are subharmonic, then so are $\max(u_1, u_2)$, and $au_1 + bu_2$ for any $a, b > 0$.

Day 20

Proposition 130 (Maximum principle)**.** *Let U be a region, $\phi : U \to \mathbb{R}$ subharmonic. If there is a point $z_0 \in U$ such that $\phi(z_0) \geq \phi(z)$ for every $z \in U$, then ϕ is a constant function.*

Proof. Let $A := \{z \in U; \phi(z) = \phi(z_0)\}$. ϕ is continuous, thus A is closed in U. Let $z_1 \in A$, $\rho > 0$ with $B(z_1, \rho) \subset U$, and suppose the existence of $\omega \in B(z_1, \rho) \cap A^C$. Thus, there exists V neighbourhood of ω with $u(z) < u(z_0)$ for any $z \in V$.
For $\omega = z_1 + re^{i\theta}$, there is a non-null subset for which $|\omega| < \phi(z_0)$. Since ϕ is subharmonic, we have:

$$\phi(z_1) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(z_1 + re^{i\tilde{\theta}}) d\tilde{\theta} < \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(z_0) d\tilde{\theta} = \phi(z_0) = \phi(z_1).$$

This is a contradiction, thus A is open and by connectedness, $A = U$. Finally, u is constant. □

Remark 131. *There exists another maximum principle in [4]. If ϕ subharmonic and ψ superharmonic, and for any $\beta \in \partial U$:*

$$\limsup_{z \to \beta} \phi(z) \leq \liminf_{z \to \beta} \psi(z),$$

then either $\phi(z) < \psi(z)$ for any $z \in U$, or $\phi(z) = \psi(z)$ and ϕ is harmonic.

Remark 132. *There is also a minimum principle for superharmonic functions.*

Definition 133 ([4])**.** *Let U be a region, $g : \partial U \to \mathbb{R}$ (thus with the topology of $\hat{\mathbb{C}}$). The Perron family $\mathcal{P}(g, U)$ consists of all subharmonic functions $\phi : U \to \mathbb{R}$ such that:*

$$\lim_{z \to \beta} \phi(z) \leq g(\beta), \quad \forall \beta \in \partial U.$$

With the topology of $\hat{\mathbb{C}}$, it also includes the case of the upper half-plan, where the boundary in $\hat{\mathbb{C}}$ is a circle.

Remark 134. *If $|g|$ is continuous on a compact set (∂U closed in a compact set $\hat{\mathbb{C}}$), then it is bounded by M. Thus $\phi : z \mapsto -M$ on U is a function in $\mathcal{P}(g, U)$, and $\mathbb{P}(g, U) \neq \emptyset$.*

Theorem 135 ([10])**.** *Let U be a region, $g : \partial U \to \mathbb{R}$ continuous. Then $u(z) := \sup\{\phi(z); \phi \in \mathcal{P}(g, U)\}$ defines a harmonic function u on U.*

Proof. $|g|$ is bounded by $M > 0$, since g is continuous on a compact set. Thus $\phi \in \mathcal{P}(g, U)$ implies $\phi(z) \leq M$ for any $z \in U$ by the maximum principle. Then u is well-defined on U.
First step: Behaviour of u around one point $z_0 \in U$. Let $z_0 \in U$, and $(\phi_n)_n \subset \mathcal{P}(g, U)$ such that $\phi_n(z_0) \to u(z_0)$. We can assume that $\phi_1 \leq \phi_2 \leq \cdots$, for any $z \in U$ by replacing ϕ_n by $\{\phi_1, \cdots, \phi_n\}$ (in some contexts, we also add $\inf_{\partial U}$, to avoid the value $-\infty$, in case the subharmonic functions are defined as semi-continuours functions).
Now, we focus on $B(z_0, R) \subset U$, for some $R > 0$. We define the functions:

$$\begin{aligned} \overline{\phi_n} : U &\to \mathbb{R} \\ z &\mapsto \begin{cases} \phi_n(z) & \text{if } z \notin B(z_0, R), \\ \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} P_r(\theta - t)\phi_n(z_0 + Re^{it})dt & \text{if } z = z_0 + re^{i\theta} \in B(z_0, R). \end{cases} \end{aligned}$$

Thus $\overline{\phi_1} \leq \overline{\phi_2} \leq \cdots$ on $\partial B(z_0, R)$, and by the maximum principle on harmonic functions (on $\overline{\phi_n} - \overline{\phi_m}$ on $B(z_0, R)$):

$$\overline{\phi_1} \leq \overline{\phi_2} \leq \cdots, \quad \text{on } B(z_0, R).$$

By Harnack's theorem, $(\overline{\phi_n})_n$ converges uniformly on $B(z_0, R)$ to ϕ, where ϕ is harmonic on $B(z_0, R)$. Furthermore:

$$\phi_n(z_0) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_n(z_0 + Re^{i\theta})d\theta = \overline{\phi_n}(z_0) \leq u(z_0),$$

and $\lim_{n \to +\infty} \phi_n(z_0) = u(z_0)$. Thus $\phi(z_0) = u(z_0)$.
Second step: The previous construction works on $B(z_0, R)$, but we claim that the function ϕ is in fact equal to u on this ball. Suppose that there exists $z_1 \in B(z_0, R)$ such that $\phi(z_1) < u(z_1)$. It implies that there exists $\tilde{u} \in \mathcal{P}(g, U)$ such that $\phi(z_1) < \tilde{u}(z_1)$. We use the same argument as before, with $w_n := \max\{\phi_n, \tilde{u}\}$. Those functions w_n are in the Perron family, and $\overline{w_n}$ converges uniformly on (compact subsets of) $B(z_0, R)$ to w, and:

$$\begin{aligned} w(z_0) = u(z_0) &= \phi(z_0), \\ \phi(z_1) \leq w(z_1) &\leq u(z_1). \end{aligned}$$

In fact, since $\overline{w_n} \geq \overline{\phi_n}$ on $B(z_0, R)$, we have $\phi \leq w$ on $B(z_0, R)$. $\phi - w$ is harmonic on $B(z_0, R)$ and non-positive; the maximum is reached $z_1 \in B(z_0, R)$. Thus $\phi = w$ by the maximum principle for harmonic functions.
This is a contradiction, because:

$$w(z) = \lim_n \overline{w_n}(z_1) = \lim_n \overline{\max\{\phi_n(z), \tilde{u}(z)\}} \geq \tilde{u}(z) > \phi(z) = w(z).$$

Thus $\phi = u$ on $B(z_0, R)$.
As a conclusion, because z_0 was chosen arbitrarly, u is harmonic on U. □

Definition 136 ([4])**.** *Let U be a region, $g : \partial U \to \mathbb{R}$ continuous. The previous function is called the Perron function associated with g.*

We want to say that this function is the solution of the Dirichlet problem. However, in some cases, the function is not the solution.

We can now give an example for which the Perron function is not a solution, the one on the punctured disk $U := B(0, 1)_{\setminus\{0\}}$, and:

$$\begin{aligned} g : \partial U &\to \mathbb{C}, \\ z &\mapsto \begin{cases} 0 \text{ if } |z| = 1, \\ 1 \text{ if } z = 0. \end{cases} \end{aligned}$$

For any $\epsilon > 0$, consider $u_\epsilon(z) := \frac{1}{\ln(\epsilon)} \ln(|z|)$. u_ϵ is harmonic on U (recall that $|z|^2 = z\bar{z}$). $u_\epsilon > 0$, $u_\epsilon(z) = 0$ for $|z| = 1$.
Suppose that $v \in \mathcal{P}(g, U)$. By the maximum principle, $v(z) \leq 1$ on U, because $|g| \leq 1$. For $\mathcal{A}_{\epsilon,1} := \{|z| \in (\epsilon, 1)\}$, we have, since $u_\epsilon = 1$ on $|z| = \epsilon$:

$$\lim_{z \to \beta} v(z) \leq u_\epsilon(\beta) = 1, \quad \forall |\beta| = \epsilon.$$

By the maximum principle (for subharmonic functions):

$$v(z) \leq u_\epsilon(z), \qquad \text{for any } z \in \mathcal{A}_{\epsilon,1}.$$

It gives in particular that $v(z) \leq \lim_{\epsilon \to 0} u_\epsilon(z) = 0$. Thus $v \leq 0$ on U, and the Dirichlet problem can not be solved on the punctured disk.
The same problem occurs by studying the set of superharmonic functions; we thus have $v \geq 0$.
(To know more about conditions of the boundary, Dirichlet domains and the notion of barrier for which the Perron function is the unique solution, see [10])

9 Product expansions [17]

9.1 Inifinite products

Definition 137. *Let $(a_n)_n \in \mathbb{C}^{\mathbb{N}}$ be a sequence of complex numbers. The product $\prod_{n=1}^{\infty}(1+a_n)$ converges if the sequence:*

$$\prod_{k=1}^{n}(1+a_k)$$

converges as $n \to +\infty$.

Proposition 138. *If $\sum_n |a_n|$ converges, then $\prod_{n=1}^{+\infty}(1+a_n)$ converges. Furthermore, $\prod_{n=1}^{+\infty}(1+a_n) = 0$ if and only if $a_n = -1$ for some n.*

Proof. We want to use the function Log : we thus need to be close enough to 1. Let $N \geq 0$ such that for any $n \geq N$, $|a_n| \leq \frac{1}{2}$. Thus $Log(1+a_n)$ is well-defined for those n (we can now forget the first N terms). We have:

$$\prod_{n=N}^{k}(1+a_n) = \exp\left(\sum_{n=N}^{k} Log(1+a_n)\right), \qquad \text{with } |Log(1+a_n)| < 2|a_n|,$$

and thus $\sum_{n=N} Log(1+a_n)$ tends to a limit l as k goes to $+\infty$. By continuity of the exponential, $\prod_{n=N}^{k}(1+a_n)$ tends to e^l.
Furthermore, if $(1+a_n) \neq 0$ for all n, then the product converges to a non-null limit. □

For example, the infinite product $\prod_{n=1}^{+\infty}\left(1+\frac{1}{n^2}\right)$ converges.

Proposition 139. *Let U be a region, $(F_n)_n \in H(U)^{\mathbb{N}}$. Suppose that:*

$$\exists c_n > 0, \quad \sum_n c_n < \infty, \quad \text{and} \quad \forall z \in U, \quad |F_n(z) - 1| \leq c_n.$$

We thus have:

- *$\prod_{n=1}^{+\infty} F_n(z)$ converges uniformly in U to $F \in H(U)$.*
- *If $F_n(z)$ does not vanish for any n, then:*

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{+\infty}\frac{F_n'(z)}{F_n(z)}.$$

Proof. For the first part, we write $F_n(z) := 1 + a_n(z)$. Thus, for any z, $F(z) := \prod_n F_n(z)$ is well-defined by the previous proposition. Also, $\left(\prod_{n=1}^{N} F_n(z)\right)_N$ converges uniformly to F on any compact subset K of U, since:

$$\left|\sum_{n=N}^{+\infty} Log(1+a_n(z))\right| \leq \sum_n c_n < \infty, \qquad |a_n(z)| < c_n$$

and the infinite sum converges uniformly, and by continuity of the exponential, $\prod_{n\geq N} F_n$ converges uniformly on K.
For the second point, let $G_n(z) := \prod_{n=1}^{N} F_n(z)$, on K compact subset of U. Then G_N' converges uniformly on K to F'. G_N is uniformly bounded from below on K, thus $\frac{G_N'}{G_N}$ is well-defined, and $\frac{G_N'}{G_N}$ tends to $\frac{F'}{F}$. Since K is arbitrary, the convergence holds on U. Furthermore,

$$\frac{G_N'(z)}{G_N(z)} = \frac{1}{G_N(z)}\left(\sum_{n=0}^{N} F_n'(z)\prod_{k\neq n} F_k(z)\right) = \sum_{n=0}^{N}\frac{F_n'(z)}{F_n(z)}.$$

□

For example, consider the function $f(z) = \frac{\sin(\pi z)}{\pi}$. What is its infinite product? By the previous decomposition, we need to evaluate the zeros of this function.
Notice that a factorization has to show the different zeros of the function sin, which are already known (by the way, what is the function sin restricted to the imaginary axis?)
By computing the derivative and the logarithmic derivative where it is possible:

$$f'(z) = \cos(\pi z), \quad \frac{f'(z)}{f(z)} = \pi \cot(\pi z).$$

Let $\omega \in \mathbb{C} \backslash \mathbb{Z}$, and for any $N > 0$, the contour $\Gamma_N := \mathcal{C}(0, N + \frac{1}{2})$. We have:

$$\int_{\Gamma_N} \frac{\pi \cot(\pi z)}{(\omega + z)^2} dz = \int_{\theta=0}^{2\pi} \frac{\pi \cot(\pi(N + \frac{1}{2})e^{i\theta})}{(\omega + (N + \frac{1}{2})e^{i\theta})^2} i \left(N + \frac{1}{2}\right) e^{i\theta} d\theta.$$

The singularities are a pole of order 2 at $-\omega$ and poles of order 1 at $\mathbb{Z}$.
What are the residues? For any $n \in \mathbb{Z}$, and z close to n:

$$\frac{\sin(\pi z)}{z - n} = \frac{\sin(\pi(z - n)) \cos(\pi n)}{z - n} \underset{z \to n}{\to} \pi(-1)^n,$$

thus $\pi \cot(\pi z) = \frac{1}{z-n} + O_{z \to n}(1)$.
By the residue formula, and that $\cot(\pi z)$ is bounded on Γ_N independently of N:

$$0 \underset{N \to +\infty}{\leftarrow} \int_{\Gamma_N} \frac{\pi \cot(\pi z)}{(\omega + z)^2} dz = 2\pi i \left(\sum_{|n| \leq N + \frac{1}{2}} \frac{1}{(n + \omega)^2} + \pi^2 \frac{-\cos(\pi\omega)^2 - \sin^2(\pi\omega)}{\sin(\pi\omega)^2} \right),$$

and so:

$$\frac{\pi^2}{\sin(\pi z)^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(n + \omega)^2},$$

and the function $\tilde{f}(z) := \pi \cot(\pi z) + \sum_{n \in \mathbb{Z}} \frac{1}{n+z}$ is in $H(\mathbb{C} \backslash \mathbb{Z})$, with a null derivative, and all the singularities are removable. Notice that the sum is not well-defined, we need to understand the sum in the following sense:

$$\sum_{n \in \mathbb{Z}} \frac{1}{n + z} = \frac{1}{z} + \sum_{n \in \mathbb{N}} \frac{2z}{z^2 - n^2}.$$

By noticing that $\tilde{f}(0) = 0$, the function is equal to 0 everywhere, and we have on $\mathbb{C} \backslash \mathbb{Z}$:

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n \in \mathbb{N}} \frac{2z}{z^2 - n^2}.$$

Thus, up to a constant, we finally find:

$$\frac{\sin(\pi z)}{\pi} = Cz \prod_{n=1}^{+\infty} \left(1 - \frac{z^2}{n^2}\right).$$

By an asymptotic development around 0, we thus have $C = 1$.

9.2 Weierstrass infinite product [17]

In this part, we choose a sequence of zeros, and try to find a function whose zeros are exactly those points.

Theorem 140. *Let $(a_n)_n \in \mathbb{C}^{\mathbb{N}}$, with $|a_n|$ increasing and tending to $+\infty$. There exists an entire funtion f that vanishes at all a_n and nowhere else. Any other function is of the form $f(z)e^{g(z)}$, with g an entire function.*

Remark 141. *The multiplicity is taken into account, by repeating the value a_n if necessary: if a is a zero of order m, then $f(z) = (z - a)^m g(z)$, with $g(a) \neq 0$.*

Proof. If f_1 and f_2 satisfy the assumption, on $|z| \leq |a_N|$ for any n, the two following functions do not vanish:

$$\frac{f_1}{\prod_{n \leq N}(z - a_n)} \quad \text{and} \quad \frac{f_2}{\prod_{n \leq N}(z - a_n)}.$$

The function $\frac{f_1}{f_2}$ has a removable singularities on (any compact of) $\mathbb{C}$, and never vanishes; there exists a logarithmic, or in other words, a function g satisfying $\frac{f_1}{f_2} = e^g$.
We now need to prove the first part. As for the function sin, we could think of a decomposition with a product $\left(1 - \frac{z^2}{n^2}\right)$, and induces the factor $\left(1 - \frac{z}{a_n}\right)$. The issue is that we do not have any clue about the convergence of this serie, since it would hold only for adequate values of a_n. To this aim, we introduce new "basic brics" for a factorization, which include the factors $(1 - \frac{z}{a_n})$:

Definition 142. *The canonical factors are defined for $k \geq 0$ by:*

$$E_0(z) = 1 - z; \qquad \textit{and } \forall k \geq 1, E_k(z) := (1 - z)e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}.$$

Those factors recall an asymptotic development of *Log*:

Lemma 143. *If $|z| \leq \frac{1}{2}$, for a constant C independent of k, we have:*

$$|1 - E_k(z)| \leq C|z|^{k+1}.$$

Proof. If $|z| \leq \frac{1}{2}$:

$$E_k(z) = \exp\left(Log(1 - z) + z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right) \leq C|z|^{k+1}.$$

□

We now define the Weierstrass product, with m the number of $a_n = 0$:

$$f(z) = z^m \prod_{n=0}^{+\infty} E_n\left(\frac{z}{a_{m+n}}\right).$$

Does this function satisfy the theorem? Let $R > 0$.

- f is well-defined on $B(0, R)$: let $N > 0$ such that for any $n \geq N$, $|a_n| \geq 2R$. By the previous lemma:

$$\left|\frac{z}{a_n}\right| \leq \frac{1}{2}, \quad \left|1 - E_n\left(\frac{z}{a_n}\right)\right| \leq C\left|\frac{z}{a_n}\right|^{n+1} \leq \frac{C}{2^{n+1}}.$$

  Thus for any R, we have $f \in H(B(0, R))$.

- f is entier, and 0 is a zero of order m.

□

Day 22

9.3 Hadamard factorization theorem

Definition 144. *An entire function $f \in H(\mathbb{C})$ has an order of growth lower than ρ if there exists two positive constants A and B such that:*

$$\forall z, \quad |f(z)| \leq Ae^{B|z|^\rho} = A\exp\left(B|z|^\rho\right).$$

$\rho_0 := \inf \rho$ *is the order of growth of f, where ρ satisfies the previous inequality.*

For example, the order of growth of a polynomial is 0. We remark that the order of growth may not bound the function at infinity.
The order of growth of the function sin is $\rho_0 = 1$ (imaginary axis, for example).
Is there any holomorphic which does not have an order of growth? Yes, as $f(z) = \exp(\exp(z))$. In the later, we study functions with a finite order of growth. It does not cover the all set of holomorphic functions, but it aready covers a large set of functions.
We denote by $n(r)$ the number of zeros of a certain function f inside $D(0, r)$.

Lemma 145. *If $z_1, \cdots, z_N$ are the (non-zero) zeros of a function f in the disk $D(0, R)$, then:*

$$\int_0^R n(r)\frac{dr}{r} = \sum_{k=1}^{N} \ln\left|\frac{R}{z_k}\right|.$$

Proof. We denote the functions:

$$\eta_k(r) := \begin{cases} 1 \text{ if } |z_k| < r, \\ 0 \text{ if } |z_k| \geq r. \end{cases} \tag{14}$$

Those functions count the number of zeros:

$$\forall r < R, \quad n(r) := \sum_{k=1}^{N} \eta_k(r).$$

Thus, we have:

$$\begin{aligned}
\sum_{k=1}^{N} \ln\left|\frac{R}{z_k}\right| &= \sum_{k=1}^{N} \int_{1}^{\left|\frac{R}{z_k}\right|} \frac{1}{r} dr \\
&= \sum_{k=1}^{N} \int_{|z_k|}^{R} \frac{dr}{r} = \sum_{k=1}^{N} \int_{0}^{R} \frac{\eta_k(r)}{r} dr \\
&= \int_{0}^{r} \frac{1}{r} \left(\sum_{k=1}^{N} \eta_k(r) \right) dr = \int_{0}^{R} n(r) \frac{dr}{r}.
\end{aligned}$$

□

Theorem 146. *If $f \in H(\mathbb{C})$ has an order of growth $\rho_0 \geq 0$,then ;*

- *$n(r) \leq Cr^{\rho}$, for some $C > 0$ and r large enough, for any $\rho > \rho_0$.*
- *If $z_1, z_2, \cdots$ are the zeros of f, with $z_k \neq 0$, we have for any $s > \rho_0$:*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty.$$

Remark 147. *The zeros are countable, do you see why? For any $R > 0$, there is a finite number of zeros in $\overline{B(0,R)}$.*

Proof. Without loss of generality, we suppose $f(0) \neq 0$. Thus, by the Jensen formula:

$$\int_0^R n(x) \frac{dx}{x} = \sum_{k=1}^{N} \ln\left|\frac{R}{z_k}\right| = \frac{1}{2\pi} \int_0^{2\pi} \ln\left|f(Re^{i\theta})\right| d\theta - \ln|f(0)|.$$

Let $R = 2r$. On one hand:

$$\int_r^{2r} n(x) \frac{dx}{x} \leq \frac{1}{2\pi} \int_0^{2\pi} \ln\left|f(Re^{i\theta})\right| d\theta - \ln|f(0)|.$$

On the other hand, $n(x)$ increases with x:

$$\int_r^{2r} n(x) \frac{dx}{x} \geq n(r) \int_r^{2r} \frac{dx}{x} = n(r)\ln(2).$$

By the growth condition of f, with $\rho > \rho_0$:

$$\frac{1}{2\pi} \int_0^{2\pi} \ln\left|f(Re^{i\theta})\right| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \ln\left(Ae^{BR^{\rho}}\right) \leq CR^{\rho}.$$

It implies $n(r) \leq Cr^{\rho}$, for r large enough, and C a constant.
To prove the second point, with $s > \rho > \rho_0$:

$$\begin{aligned}
\sum_{|z_k| \geq 1} \frac{1}{|z_k|^s} &= \sum_{j=0}^{+\infty} \left(\sum_{2^j \leq |z_k| < 2^{j+1}} \frac{1}{|z_k|^s} \right) \\
&\leq \sum_{j=0}^{+\infty} \frac{n(2^{j+1})}{2^{js}} \leq C \sum_j \frac{2^{j\rho}}{2^{js}} = C \sum_j \frac{1}{2^{js}} < \infty.
\end{aligned}$$

□

As an application, the function $f(z)=\sin(\pi z)$ satisfies $|f(z)|\leq ce^{\pi|z|}$, and its order of growth is $\rho_0=1$; since the set of zeros is $\mathcal{Z}(f)=\mathbb{Z}$, we have, for any $s>1$:

$$\sum_{n\in\mathbb{Z}^*}\frac{1}{n^s}<\infty.$$

Theorem 148 (Hadamard)**.** *Suppose that $f\in H(\mathbb{C})$ has a order of growth ρ_0. Let $k\in\mathbb{N}$, such that $k\leq\rho_0<k+1$. If $z_0,z_1,\cdots,z_n,\cdots$ are the (non-zero) zeros of f, then:*

$$f(z)=e^{P(z)}z^m\prod_{n=1}^{\infty}E_k\left(\frac{z}{z_n}\right),$$

where P is the polynomial of degree $\leq k$ and m is the order of the zero of f at $z=0$.

Remark 149. *According to the growth of f, we use different "elementary bricks" E_k.*

Remark 150. *Notice the difference with the previous paragraph : we were considering a set of zeros without space localization, and it implies the use of different canonical factors E_k. In the Hadamard theorem, we know the order of growth, and thus an estimate on the distribution of zeros. We only use one canonical factor E_k dependent on this growth.*

Lemma 151. *The two following estimates on the canonical factors hold:*

$$\forall|z|\leq\frac{1}{2},\quad |E_k(z)|\geq e^{-c|z|^{k+1}},$$
$$\forall|z|\geq\frac{1}{2},\quad |E_k(z)|\geq|1-z|e^{-c'|z|^k}.$$

Proof. If $|z|\leq\frac{1}{2}$:

$$\left|Log(1-z)+\sum_{k=1}^{K}\frac{z^k}{k}\right|\leq c|z|^{K+1},$$
$$|E_K(z)|\geq\exp\left(-\left|Log(1-z)+\sum_{k=1}^{K}\frac{z^k}{k}\right|\right)\geq e^{-c|z|^{K+1}}.$$

The same strategy holds for the second inequality. ☐

Lemma 152. *Let $s\in(\rho_0,k+1)$. The following inequality holds:*

$$\forall z\in\left(\bigcup_{n=0}^{+\infty}B\left(z_n,\frac{1}{|z_n|^{k+1}}\right)\right)^C,\quad\left|\prod_{n=1}^{+\infty}E_k\left(\frac{z}{z_n}\right)\right|\geq e^{-c|z|^s}.$$

Proof. We cut the product into two pieces, for a z in the previous domain:

$$\prod_{n=1}^{+\infty}E_k\left(\frac{z}{z_n}\right)=\left(\prod_{|z_n|\leq2|z|}E_k\left(\frac{z}{z_n}\right)\right)\left(\prod_{|z_n|>2|z|}E_k\left(\frac{z}{z_n}\right)\right)=(A)(B).$$

For the term (B), by the previous lemma:

$$|(B)|=\prod_{|z_n>2|z|}\left|E_k(\frac{z}{z_n})\right|\geq\prod_{|z_n|>2|z|}\exp\left(-c\left|\frac{z}{z_n}\right|^{k+1}\right)$$
$$\geq\exp\left(-c|z|^{k+1}\sum_{|z_n|>2|z|}\frac{1}{|z_n|^{k+1}}\right),$$

and since $s<k+1$:

$$\sum_{|z_n|>2|z|}\frac{1}{|z_n|^{k+1}}\leq C\sum_{|z_n|>2|z|}\frac{1}{|z_n|^s|z|^{k+1-s}}\leq\frac{c}{|z|^{k+1-s}},$$

and thus

$$|(B)| \geq c \exp\left(-c|z|^s\right).$$

For (A), by the other inequality of the previous lemma:

$$\begin{aligned}
|(A)| &\geq \prod_{|z_n|\leq 2|z|} \left|1 - \frac{z}{z_n}\right| \prod_{|z_n|\leq 2|z|} \exp\left(-c' \left|\frac{z}{z_n}\right|^k\right) \\
&\geq \prod_{|z_n|\leq 2|z|} \left|1 - \frac{z}{z_n}\right| \exp\left(-c' \sum_{|z_n|\leq 2|z|} \left|\frac{z}{z_n}\right|^k\right),
\end{aligned}$$

and since $k - s < 0$:

$$\frac{1}{|z_n|^k} \leq \frac{1}{|z_n|^s} \frac{1}{|z|^{k-s}},$$

and thus

$$\exp\left(-c' \sum_{|z_n|\leq 2|z|} \left|\frac{z}{z_n}\right|^k\right) \geq e^{-c|z|^s}.$$

Now, with the assumption on $z \notin \cup_n B(z_n, |z_n|^{-k-1})$:

$$\begin{aligned}
\prod_{|z_n|>2||} \left|1 - \frac{z}{z_n}\right| &= \prod_{|z_n|\leq 2|z|} \left|\frac{z_n - z}{z_n}\right| \\
&\geq \prod_{|z_n|\leq 2|z|} \frac{1}{|z_n||z_n|^{k+1}} = \prod_{|z_n|\leq 2|z|} \frac{1}{|z_n|^{k+2}}.
\end{aligned}$$

It implies, by the previous theorem and $\rho_0 < \rho < s$:

$$\begin{aligned}
\prod_{|z_n|\leq 2|z|} \frac{1}{|z_n|^{k+2}} &= \exp\left(-(k+2) \sum_{|z_n|\leq 2|z|} \ln|z_n|\right) \\
&\geq \exp\left(-(k+2)n(|z|)\ln(2|z|)\right) \\
&\geq \exp\left(-(k+2)c|z|^\rho \ln(2|z|)\right) \\
&\geq \exp\left(-c|z|^s\right).
\end{aligned}$$

□

Day 23

Lemma 153. *There exists an increasing sequence $(r_m)_m \in \mathbb{R}_+^{\mathbb{N}}$ tending to $+\infty$, such that:*

$$\forall |z| = r_m, \quad \left|\prod_{n=1}^{+\infty} E_k\left(\frac{z}{z_n}\right)\right| \geq e^{-c|z|^s}.$$

This proof justifies that the set of the previous lemma is not empty! It is in fact quite large.

Proof. By the previous theorem, $\sum |z_n|^{-k-1} < \infty$, so there exists $N > 0$ such that $\sum_{n\geq N} |z_n|^{-k-1} < \frac{1}{2}$. Let $L \in \mathbb{N}$, large enough. We claim that there exists $r \in [L : L+1]$, such that $\partial D(0,r) = \mathcal{C}(0,r)$ does not intersect any forbidden disk. If it was not the case, the union of intervals:

$$I_n := \left[|z_n| - \frac{1}{|z_n|^{k+1}}; |a_n| + \frac{1}{|z_n|^{k+1}}\right]$$

would cover the all interval $[L; L+1]$. Thus $2\sum_{n\geq N} |a_n|^{k+1} > 1$ which is a contradiction. □

(see also the nice pictures in [17]).

Proof of the Hadamard factorization theorem. Let $E(z) := z^m \prod_{m=1}^{+\infty} E_k\left(\frac{z}{z_n}\right)$. E is well-defined because, for $\rho_0 < s < k+1$:

$$\forall z \in \mathbb{C}, \forall |z_n| \geq 2|z|, \quad \left|1 - E_k\left(\frac{z}{z_n}\right)\right| \leq C \left|\frac{z}{z_n}\right|^{k+1} \quad \text{and} \quad \sum_n \frac{1}{|z_n|^{k+1}} < \infty.$$

Moreover, E has the zeros of f : $\frac{f}{E}$ is holomorphic (removable singularities) and never vanishes. Thus, there exists a logarithm g, satisfying $\frac{f}{E} = e^g$. Furthermore, if $|z| = r_m$ with the previous lemma, and $\rho < s$:

$$\exp\left(\mathrm{Re}(g)(z)\right) = \left|\frac{f(z)}{E(z)}\right| \leq \frac{c'_f e^{-c'_f |z|^\rho}}{c'_E e^{-c_E |z|^s}} \leq c' e^{c|z|^s},$$

which proves that $\mathrm{Re}(g)(z) \leq c|z|^s$ for $|z| = r_m$. We thus admit the following lemma to conclude the proof:

Lemma 154. *If $g \in H(\mathbb{C})$ and $u := Re(g)$ satisfies $u(z) \leq cr^s$ for $|z| = r$, for a sequence $r \to +\infty$, then g is a polynomial of degree at most s.*

This lemma is sufficient to conclude the proof of the Hadamard theorem. □

Proof. Proof of the previous lemma Let $g \in H(\mathbb{C})$, $g(z) = \sum_{n=0}^{+\infty} b_n z^n$, with $b_n \in \mathbb{C}$. We have:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(re^{i\theta}) e^{-in\theta} d\theta = \begin{cases} b_n r^n, \text{ if } n \geq 0, \\ 0, \text{ if } n < 0. \end{cases}$$

If $n > 0$, we have:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{g(re^{i\theta})} e^{-in\theta} d\theta = 0.$$

With $2u = g + \overline{g}$, for any $n \geq 0$, we have:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) e^{-in\theta} d\theta b_n r^n.$$

We can thus compare u with the polynomial, for any $n > s$:

$$\begin{aligned} b_n &= \frac{1}{\pi r^n} \int_{-\pi}^{\pi} \left(u(re^{i\theta}) - Cr^s\right) e^{-in\theta} d\theta, \\ |b_n| &\leq \frac{1}{\pi r^n} \int_{-\pi}^{\pi} \left|Cr^s - u(re^{i\theta})\right| d\theta \\ &\leq 2Cr^{s-n} \underset{r \to +\infty}{\to} 0. \end{aligned}$$

□

Let us come back on the factorization of the function $\frac{\sin(\pi z)}{\pi}$. Its order of growth is 1. By the Hadamard factorization theorem, there exists a polynomial $P \in \mathbb{C}[X]$, with $deg(P) \leq 1$, so that:

$$\begin{aligned} \frac{\sin(\pi z)}{\pi} &= e^{P(z)} z \prod_{n \neq 0} E_1\left(\frac{z}{n}\right), \\ &= e^{P(z)} z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} = e^{P(z)} z \prod_{n \in \mathbb{N}} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right) \\ &= e^{P(z)} z \prod_{n \in \mathbb{N}} \left(1 - \frac{z^2}{n2}\right). \end{aligned}$$

We need now to find the coefficients of P. $\frac{\sin(\pi z)}{\pi z} \to 1$ as $z \to 0$, thus $e^{P(0)} = 1$, $P(0) = 0[2\pi]$. We can choose $P(z) = az$.
To find the value of a, we use the formula of the logarithmic derivative of an infinite product, for $z \in \mathbb{C} \backslash \mathbb{Z}$:

$$\pi \frac{\cos(\pi z)}{\sin(\pi z)} = P'(z) + \frac{1}{z} + \sum_{n \in \mathbb{N}} \frac{2z}{z^2 - n^2}.$$

For $N \geq 0$:

$$f_N(z) := \frac{1}{z} + \sum_{n \leq N} \frac{2z}{z^2 - n^2} = \frac{1}{z} + \sum_{n \leq N} \left(\frac{1}{z-n} + \frac{1}{z+n} \right),$$

and thus

$$f_N' \frac{1}{2}) = 2 + \sum_{n \leq N} \left(\frac{2}{1-2n} + \frac{2}{1+2n} \right) = \frac{2}{1+2N},$$

so $\lim_N F_N(\frac{1}{2}) = 0 = -P'(\frac{1}{2})$, so $a = 0$. Finally we get:

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n \in \mathbb{N}} \left(1 - \frac{z^2}{n^2} \right).$$

10 Introduction to elliptic functions

The elliptic functions are used in algebra, and number theory. Also, there are useful tools to compute elliptic integral (for example, a length of an arc of an ellipse.)

10.1 Lattices [17, 1]

Definition 155. *An elliptic function f is a non-constant meromorphic function on $\mathbb{C}$ which is doubly periodic : there exist $\omega_1 \in \mathbb{C}^*$ and $\omega_2 \in \mathbb{C}^*$, with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ such that:*

$$\forall z \in \mathbb{C}, \quad f(z+\omega_1) = f(z) = f(z+\omega_2).$$

The condition that ω_1 and ω_2 are $\mathbb{R}$-linear independent avoids the case of combinations of ω_1 and ω_2 on one line.
By the definition, any combination (over $\mathbb{Z}$) of ω_1 and ω_2 will satisfy:

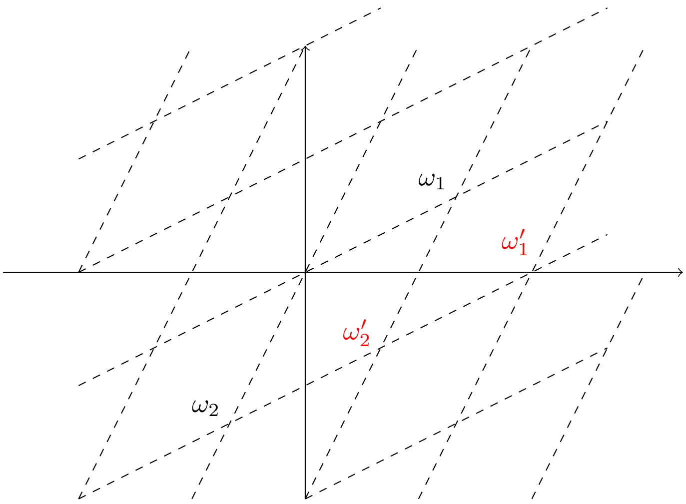
$$\forall n, m \in \mathbb{Z}, \forall z \in \mathbb{C}, \quad f(z+n\omega_1+m\omega_2) = f(z).$$

Let us delve in the different possibilities of ω_1 and ω_2.

Definition 156. *If the complex numbers ω_1 and ω_2 are $\mathbb{R}$-linear independent, then the following set is a lattice in $\mathbb{C}$:*

$$\Lambda := \left\{n\omega_1 + m\omega_2; (n,m) \in \mathbb{Z}^2\right\}.$$

In this case, we say that ω_1 and ω_2 generate the lattice Λ.
Naturally, we denote Λ^* the set Λ without 0.



On this picture, the intersection points of the dashed lines are the points of the lattice.

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Definition 157. *If Λ is generated by ω_1 and ω_2, then (ω_1, ω_2) is a basis of Λ. A fundamental parallelogram associated to the lattice Λ is defined by:*

$$P_\omega := \{z \in \mathbb{C}; z = \omega + a\omega_1 + b\omega_2; 0 \le a < 1; 0 \le b < 1\},$$

where (ω_1, ω_2) is a basis.
In the previous example, there are two bases of the lattice : (ω_1, ω_2) and (ω_1', ω_2').
Let us now investigate the different bases ([1]). If (ω_1, ω_2) and (ω_1', ω_2') are two bases, then there exists integers (a, b, c, d) such that:

$$\omega_1 = a\omega_1' + b\omega_2',$$
$$\omega_2 = c\omega_1' + d\omega_2'.$$

Since there exists a unique solution to this system, it is required that $ad - bc \neq 0$.

The action group which sends one basis to another is generated by:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \text{with } ad - bc \neq 0.$$

Similarly, to go from the base (ω_1', ω_2') to the base (ω_1, ω_2):

$$\begin{cases} \omega_1' = a'\omega_1 + b'\omega_2 \\ \omega_2' = c'\omega_1 + d'\omega_2 \end{cases},$$

or in terms of matrices, since the ω_1' and ω_2' form a basis, we obtain:

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

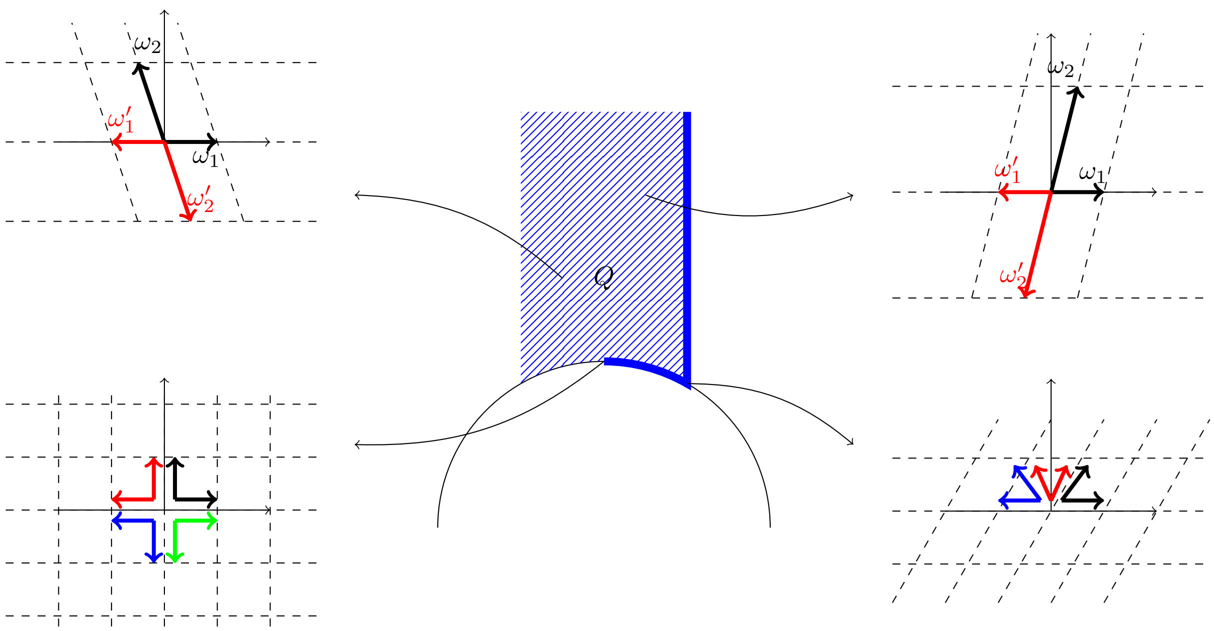
Thus the determinant of the product is equal to 1, and each matrix has a determinant equal to ± 1.
The group $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; ad - bc = \pm 1 \right\}$ is called the modular group (or equivalently, we can consider the set of applications associated to those matrices, acting on the set of bases).

Theorem 158 ([1])**.** *For each lattice Λ, there exists a basis (ω_1, ω_2) such that the ratio $\tau = \frac{\omega_1}{\omega_2}$ satisfies:*

- *$Im(\tau) > 0$,*
- *$Re(\tau) \in (-\frac{1}{2}; \frac{1}{2}]$,*
- *$|\tau| \geq 1$,*
- *if $|\tau| = 1$, then $Re(\tau) \geq 0$.*

τ is uniquely determined by those conditions, and there is a choice of two, four or six bases corresponding to these ratios.

We admit the proof of this theorem, since we have already dealt with fastidious proofs with similar ideas on the modular group on the Little Picard theorem. What is more important to get is an idea of the different possibilities of bases according to the ratio.
Let us denote by Q the set satisfying those four conditions. We thus have:



(up to rotations, $\tau = \frac{R(\omega_2)}{R(\omega_1)} = \frac{\omega_2}{\omega_1}$, thus we can consider that ω_1 is situated on the real line).
Notice that, the area of a fundamental parallelogram does not change by the change of basis since the transformations are modular!
Notice also that the modular group sends a basis to another, and each point in the upper half-plane (corresponding to τ) corresponds to a new basis.

For example ([1, 17], and personal examples), consider the Möbius transformations $S(z) = -\frac{1}{z}$ and $T(z) = z + 1$, and point $\tau = \frac{1}{4} + i$. We consider (non-unique, because of the sign) matrices associated to those transformations:

$$M_T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We thus have $M_T(\omega_2) = \omega_1 + \omega_2$, $M_T(\omega_1) = \omega_1$, $M_S(\omega_2) = -\omega_1$ and $M_S(\omega_1) = \omega_2$.

(Make a picture of the images of one basis by the different applications.)

You can remark that the following procedures, for a Möbius transformation f whose associated matrix is modular, and a point $\tau \in \mathbb{H}^+$:

- take the image of this point by f,
- consider a lattice and a basis (ω_1, ω_2) whose ratio gives τ; consider the basis-image of (ω_1, ω_2) by a matrix associated to f; take the ratio of this new basis,

give the same point $f(\tau)$.

10.2 Liouville theorems on elliptic functions [1, 17]

Theorem 159. *If f is an entire function and doubly periodic, then f is constant.*

Proof. The proof is a direct application of Liouville's theorem, since it is continuous on the closure of a fundamental parallelogram. □

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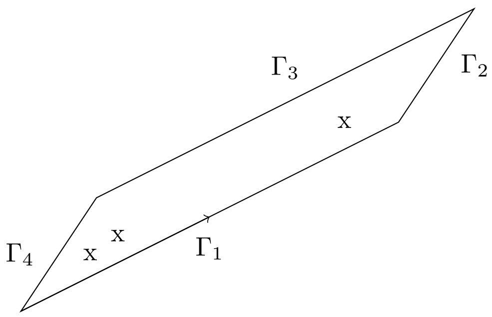
Let us denote f by an elliptic function, doubly-periodic on a lattice Λ, and (ω_1, ω_2) be a basis of Λ. With the previous theorem, f has at least one pole. The number of poles is finite on a fundamental parallelogram, since the set of poles does not have an accumulation point (definition of a meromorphic function).

Theorem 160. *The sum of the residues of an elliptic function f is zero, so f has at least two poles.*

"At least two poles" means that either there are at least two distinct poles or one pole with multiplicity at least 2.

Proof. We can choose ω so that P_ω does not have any pole of f on its boundary. We denote by γ_1, γ_2, γ_3 and γ_4 the counter-clockwise paths whose images are the sides of this parallelogram.



We thus have:

$$\int_{\partial P_\omega} f(z)dz = 0, \text{ since } \int_{\Gamma_1} f = \int_{\Gamma_3} f,$$

where the last equality comes from the doubly periodicity. Thus, by the residue theorem:

$$\sum_i \mathrm{Res}_{y_i}(f) = 0.$$

□

The <u>order</u> of an elliptic function is the number of poles in P_ω (counted with multiplicities).

Theorem 161. *By the argument principle (we suppose that ∂P_ω does not contain neither poles nor zeros of f):*

$$0 = \frac{1}{2\pi i}\int_{\partial P_\omega} \frac{f'(z)}{f(z)}dz = \sum_{i=0}^{k} m_i Ind_{\partial P_\omega}(z_i) - \sum_{j=0}^{l} n_j Ind_{\partial P_\omega}(y_j),$$

where the z_i are the zeros, y_j are the poles, and m_i and n_j their respective mutliplicity. In the previous formula, all the indices are equal to 1, and gives the final formula.

10.3 The Weierstrass functions.

10.3.1 $\wp$-function of Weierstrass [1, 17]

We have seen that an elliptic function has at least two poles. We consider one of the simplest cas, of one double pole on the points of the lattice Λ centered at 0.
How to create such a function? By definition of the poles and their order, we expect the function to be written as:

$$f(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{(z-\omega)^2} + \cdots\right).$$

However, this sum is not well-defined. Let us take example on the solution of this problem in one dimension. Since the sum $g(z) = \sum_{n\in\mathbb{Z}}(n+z)^{-1}$ is not well-defined, we add and substract the same term on different part of the sum, and we get:

$$g(z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}^*}\left(\frac{1}{n+z} - \frac{1}{n}\right) = \frac{1}{z} + \sum_{n\in\mathbb{Z}^*}\frac{-z}{(n+z)n},$$

and for a fixed z, the previous sum converges. By following the same procedure, we define our elliptic function.

Definition 162. *The $\wp$-function of Weierstrass is defined by:*

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\notin\Lambda^*}\left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right).$$

Is this function well-defined for $z \notin \Lambda$? The sum simplifies:

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{z(z-2\omega)}{(z-\omega)^2\omega^2} = O_{|\omega|\to+\infty}\left(\frac{1}{|\omega|^3}\right)$$

We claim that $\wp$ is well-defined. In fact, by the Young inequality, if $k \in \mathbb{C}\backslash\mathbb{R}$, and $(n,m) \in \mathbb{Z}$, and we choose $\epsilon \in (1; \frac{|\tau|^2}{\mathrm{Re}(\tau)^2}$:

$$\begin{aligned}|n+mk|^2 = (m\mathrm{Re}(k)+n)^2 + (m\mathrm{Im}(k))^2 &= m^2|k|^2 + 2nm\mathrm{Re}(k) + n^2 \\ &\geq m^2k^2\left(1-\frac{1}{\epsilon}\right) + n^2\left(|\tau|^2 - \epsilon\mathrm{Re}(\tau)^2\right) \geq C\left(n^2+m^2\right),\end{aligned}$$

for some consant $C > 0$. Thus the serie converges:

$$\begin{aligned}|\wp(z)| &\leq c + c'\sum_{\omega\in\Lambda^*}\frac{1}{|\omega|^3} = c + c'\sum_{\omega\in\Lambda^*,\omega=n\omega_1+m\omega_2} \\ &\leq c + c'\sum_{(n,m)\in(\mathbb{Z}^2)^*}\frac{1}{|n+mk|^3} \\ &\leq c + c''\sum_{n\in\mathbb{Z}^*}\sum_{m\in\mathbb{B}}\frac{1}{(|n|+|m|)^3} \\ &\leq c + c''\sum_{n\in\mathbb{Z}^*}\sum_{l\geq|n|}\frac{1}{l^3} \\ &\leq c + c''\sum_{n\in\mathbb{Z}^*}\frac{1}{n^2} < \infty,\end{aligned}$$

where the last inequality comes from the integral test of convergence.
We now claim that this function $\wp$ is doubly-periodic in ω_1 and ω_2. To prove that, we compute the derivative:

$$\wp'(z) = -\frac{1}{z^3} - 2\sum_{\omega\in\Lambda^*}\frac{1}{(z-\omega)^3} = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}.$$

The derivative is doubly periodic. Thus the functions $z \mapsto \wp(z) - \wp(z+\omega_1)$ and $z \mapsto \wp(z) - \wp(z+\omega_2)$ are constant. Because the function $\wp$ is even, be evaluating the functions at $\frac{-\omega_1}{2}$ and $\frac{-\omega_2}{2}$, the constants are equal to 0. We gather the previous analysis:

Theorem 163. *$\wp$ is a meromorphic function (associated with the lattice Λ), with double poles at the lattice points.*

10.3.2 Functions ζ and σ of Weierstrass [1]

(The function is different from the ζ-function of Riemann!)
We would like to find a primitive of the function $\wp$. Since there are singularities, a global primitive may not be defined : if a closed pathγ goes around one singularity, does the integral along this path give 0? It is the case, by the residue formula :

$$\frac{1}{2\pi i}\int_\gamma \wp(\omega)d\omega = \mathrm{Ind}_\gamma(z_0)\mathrm{Res}_f(z_0) = 0.$$

It thus makes sense to integrate terms by terms. The primitive would be:

$$\sum_{\omega\in\Lambda}\frac{-1}{z-\omega},$$

and once again, we need to give a sense to this function.

Definition 164. *The ζ-function of Weierstrass is defined by:*

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\neq 0}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

ζ is odd, and has simple poles of the points of the lattice.

The derivative is:

$$\zeta'(z) = -\wp(z).$$

ζ is not periodic, but there exist two values η_1 and η_2 such that:

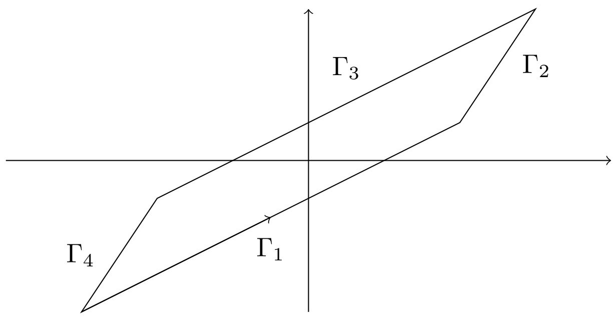
$$\forall z \notin \Lambda, \quad \begin{cases}\zeta(z+\omega_1) = \zeta(z) + \eta_1,\\ \zeta(z+\omega_2) = \zeta(z) + \eta_2.\end{cases}$$

Exercise : Find the values η_1 and η_2 in terms of the points of Λ.

Proposition 165 (Legendre relation)**.**

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

Proof. By integrating along the boundary of a fundamental parallelogram:



$$\int_{\partial P_\omega} \zeta(z)dz = 2\pi i = \int_{\Gamma_1} \zeta(z)dz + \int_{\Gamma_2} \zeta(z)dz + \int_{\Gamma_2} \zeta(z)dz + \int_{\Gamma_4} \zeta(z)dz = \int_{\Gamma_1} -\eta_2 dz + \int_{\Gamma_2} \eta_1 dz.$$

□

If we want to find a second primitive, one step further, we want to avoir the multiple-valuedness:

$$\int_0^z \zeta(w)dw \quad \text{"} = \text{"} \quad \mathrm{Log}(z) + \sum_{\omega\in\Lambda^*} \left(\mathrm{Log}(z-\omega) + \frac{z}{\omega} + \frac{z^2}{2\omega^2}\right) + c,$$

we thus take the exponential of this function, and we define:

Definition 166. *The σ-function of Weierstrass is defined by:*

$$\sigma(z) := z \prod_{\omega\in\Lambda^*} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}} = z \prod_{\omega\in\Lambda^*} E_2(\frac{z}{\omega}).$$

We recall that E_2 is a canonical factor. The function σ is entire, and well-defined since $\left|1 - E_k\left(\frac{z}{\omega}\right)\right| \leq c\left|\frac{z}{\omega}\right|^{2+1}$, which implies the convergence of the infinite product. Thus we obtain:

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z).$$

Exercise : Evaluate $\sigma(z+\omega_1)$ in terms of $\sigma(z)$.

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10.3.3 Differential equation [1]

We want to get an asymptotic expansion of ζ around 0 (in z). We thus study each term of ζ in the sum:

$$\begin{aligned} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} &= -\frac{1}{\omega}\left(\frac{1}{1-\frac{z}{\omega}}\right) + \frac{1}{\omega} + \frac{z}{\omega} \\ &= -\sum_{k=2}^{+\infty} \frac{z^k}{\omega^{k+1}}. \end{aligned}$$

Thus:

$$\zeta(z) = \frac{1}{z} - \sum_{\omega\in\Lambda^*}\sum_{k=2}^{\infty} \frac{z^k}{\omega^{k+1}} = \frac{1}{z} - \sum_{k\geq 2} G_k z^{2k-1}.$$

In fact, in the previous sum, the terms with an even power in z are not cited, because the function ζ is odd. You can prove it directly on the definition of ζ, or use the symmetry of the lattice. In fact, for any positive integer k larger than 1 :

$$\sum_{|n|\leq N, |m|\leq M} \frac{1}{(n\omega_1 + m\omega_2)^{2k+1}} = 0.$$

We can compute the different development of $\wp(z) = z^{-2} + \sum_{k\geq 2}(2k-1)G_k z^{2k-2}$ and of its derivative:

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots \\ \wp(z)^2 &= \frac{1}{z^4} + 6G_2 + 10G_3 z^2 + \cdots \\ \wp(z)^3 &= \frac{1}{z^6} + 9G_2\frac{1}{z^2} + 15G_3 + \cdots \\ \wp'(z) &= -\frac{2}{z^3} + 6G_2 z + 20G_3 z^3 + \cdots \\ (\wp'(z))^2 &= \frac{4}{z^6} - 24G_2\frac{1}{z^2} - 80G_3 + \cdots \end{aligned}$$

Proposition 167. *$\wp$ satisfies the following differential equation:*

$$(\wp'(z))^2 = 4\wp(z)^3 - 60G_2\wp(z) - 140G_3.$$

In fact, the function $z \mapsto (\wp'(z))^2 - 4\wp(z)^3 + 60G_2\wp(z) + 140G_3$ is doubly-periodic, and the poles have been cancelled. It is thus entire, and thus constant. The value of the constant has been cancelled to, and thus is equal to 0.
Generally we use the following notations:

$$g_2 := 60G_2, \quad g_3 := 140G_3,$$

so that we can write the differential equation by:

$$\boxed{\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.} \tag{15}$$

10.3.4 The modular function λ : its comeback [1, 17]

The differential equation (15) brings us to study the roots of $\wp'$ on a fundamental parallelogram. $\wp'$ is odd, periodic in ω_1 and in ω_2:

$$\wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_1}{2}\right) = -\wp'(\frac{\omega_1}{2}) = 0.$$

Similarly, $\wp'(\frac{\omega_2}{2}) = 0$ and $\wp'(\frac{\omega_1+\omega_2}{2}) = 0$. We thus denote:

$$\begin{aligned} e_1 &:= \wp\left(\frac{\omega_1}{2}\right), \\ e_2 &:= \wp\left(\frac{\omega_2}{2}\right), \\ e_3 &:= \wp\left(\frac{\omega_1+\omega_2}{2}\right). \end{aligned}$$

Notice that those values are different one from each others. In fact, each value of $\wp$ is taken only once with multiplicity 2 : the function $\wp - e_1$ is also an elliptic function, thus the number of zeros is exactly 2, and $\frac{\omega_1}{2}$ is a zero of order 2. There are no other solutions in the fundamental parallelogram of the equation $\wp(z) = e_1$ than $\frac{\omega_1}{2}$. Thus, the three values are distinct, and we have:

$$\boxed{\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)}$$

Remark 168 (no reference, own interpretation)**.** *The previous values were taken over $\mathbb{C}$. We can notice, by the previous argument, that to each point $\omega \in \mathbb{C}$, there exists exactly two points $z \in P$ satisfying the equation $\wp(z) - \omega = 0$; the two values are distinct and of multiplicity 1, except e_1, e_2 and e_3 of multiplicity 2. (Can we prove that the function is bijective from a half parallelogram, with poles on the boundary, into $\mathbb{C}$? I think we can, since the function is even.) In particular, if $\gamma : [a, b] \to \mathbb{C}$ is a path in $\mathbb{C}$, if we choose z_0 such $\wp(z_0) = \gamma(a)$ (which is not unique), there exists a path $\eta : [a, b] \to \mathbb{C}$ (not unique if its range contains e_1, e_2 and e_3, for which the derivative of $\wp$ is equal to 0) such that $\eta(a) = z_0$ and $\wp(\eta(t)) = \gamma(t)$ (Make a picture!) If you now consider a path γ in $\hat{\mathbb{C}}$ (for example a real path $(-\infty; x_0)$), and a path $\eta : [a, b] \to \mathcal{C}$ with $\gamma = \wp(\eta)$, you can compute the integral:*

$$\begin{aligned} \int_{-\infty}^{x_0} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} &= \int_\gamma \frac{d\omega}{\sqrt{4\omega^3 - g_2\omega - g_3}} \\ &= \int_\eta \frac{\wp'(z)dz}{\sqrt{4\wp(\omega)^3 - g_2\wp(z) - g_3}} \\ &= \eta(b) - \eta(a) = \text{"}\wp^{-1}(\gamma(b)) - \wp^{-1}(\gamma(a))\text{"}. \end{aligned}$$

In fact, the previous integral is the "inverse" function of the function $\wp$ (with all "" of the world). Some remarks have to be made on the previous computations:

- *the first concerns the coefficients g_2 and g_3. To define the function $\wp$, we use the serie development and the differential equation to obtain the coefficients G_k. However, if the function is defined this way, we are not sure that this serie converges. One condition is that the three coefficients, e_1, e_2 and e_3 are distinct, or, in other words, that the polynomial $P(z) = 4z^3 - g_2z - g_3$ has three different roots. The three roots of a polynomial of degree three are distinct if and only if the discriminant is equal to 0, which in our case corresponds to $\Delta = g_2^3 - 27g_3^2 \neq 0$. If this condition is satisfied, then the $\wp$-function is defined by a serie, and corresponds to a certain lattice. (this is the uniformization theorem to identify this lattice, see Theorem 2.9, in[2]) Thus the change of variable $w = \wp(z)$ is justified, and the function $\wp$ depends on the coefficients g_2 and g_3.*

- *For the last line, we used (in the wrong way!) the function $\wp^{-1}$. We can restrict the domain of $\wp$ to a fundamental parallelogram, so that the function is well defined. However, the path η may not be restricted to one parallelogram, and thus the values of $\eta(b)$ and $\eta(a)$ may not stay in this parallelogram. Furthermore, it depends on the beginning point of the chosen path η. With your knowledge, you know that we can use an analytic continuation so at each point of γ, the values of $\wp^{-1}(\gamma)$ is well-defined.*

Definition 169 (or Theorem)**.** *The function*

$$\lambda(\tau) = \lambda\left(\frac{\omega_2}{\omega_1}\right) = \frac{e_3 - e_2}{e_1 - e_2}$$

is well-defined.

Notice that three values are not taken by the function λ : 0, 1 and ∞, because e_1, e_2 and e_3 are different one from each other.

Proof. Consider two different bases (ω_1, ω_2) and (ω_1', ω_2') such that $\frac{\omega_2}{\omega_1} = \frac{\omega_2'}{\omega_1'}$. The two bases may correspond to two different lattices, and thus the $\wp$-function of Weierstrass will be different according to which lattice Λ or Λ' we consider.
There exists a complex number $t \in \mathbb{C}$ such that:

$$\omega_1 = t\omega_1', \quad \omega_2 = t\omega_2'.$$

The complex numbers ω_1, ω_2, ω_1' and ω_2' have to be considered as vectors of bases, whereas the complex number t is interpreted as a transformation, a composition of rotation and dilation which keeps invariant the ratio t. We thus get:

$$\wp_{\Lambda'}\left(\frac{\omega_1'}{2}\right) = \wp_{\Lambda'}\left(\frac{1}{t}\frac{\omega_1}{2}\right) = \frac{1}{t^2}\wp_{\Lambda}\left(\frac{\omega_1}{2}\right),$$

where the last equality comes from the definition of the $\wp$-function as a sum of quadratic terms. By this transformation, all the points in the lattice Λ' will be one-to-one exchanged with the points of Λ. $\square$

We now study how the function λ is transformed under the group action induced by the modular group. In other words, we consider a function in the modular group:

$$f : z \mapsto \frac{az + b}{cz + d} \in Aut(\hat{\mathbb{C}}), \qquad \text{with } ad - bc = 1,$$

and compare $\lambda(f(\tau))$ and $\lambda(\tau)$. Notice that from τ and $f(\tau)$, we can consider the same lattice (since f corresponds to a change of basis), and thus the same $\wp$-function associated to this unique lattice. Since the modular group is generated by the functions:

$$T : z \mapsto z + 1 \quad \text{and} \quad S : z \mapsto -\frac{1}{z},$$

it is sufficient to study how those two generators change the function λ. The matrix associated to T is $M_T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and we have the following change of basis:

$$\begin{cases} \omega_2' = 1\omega_2 + 1\omega_1 \\ \omega_1' = 0\omega_2 + 1\omega_1. \end{cases} \tag{16}$$

We now investigate how the values e_1, e_2 and e_3 defined by the basis (ω_1, ω_2) are changed by thius change of variables:

$$\begin{aligned} \wp\left(\frac{\omega_1'}{2}\right) &= \wp\left(\frac{\omega_1}{2}\right) = e_1, \\ \wp\left(\frac{\omega_2'}{2}\right) &= \wp\left(\frac{\omega_1 + \omega_2}{2}\right) = e_3, \\ \wp\left(\frac{\omega_1' + \omega_2'}{2}\right) &= \wp\left(\omega_1 + \frac{\omega_2'}{2}\right) = e_2. \end{aligned}$$

Thus, we have:

$$\lambda(T(\tau)) = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\lambda(\tau)}{\lambda(\tau) - 1}.$$

Similarly for the application S:

$$S(\tau) = S\left(\frac{\omega_2}{\omega_1}\right) = -\frac{1}{\frac{\omega_2}{\omega_1}} = \frac{-\omega_1}{\omega_2} = \frac{\omega_2'}{\omega_1'}.$$

With $\omega_1' = \omega_2$ and $\omega_2' = \omega_1$, we get:

$$\wp\left(\frac{\omega_1}{2}\right) = \wp\left(\frac{\omega_2}{2}\right) = e_2, \quad \wp\left(\frac{\omega_2'}{2}\right) = \wp\left(\frac{\omega_1'}{2}\right) = e_1.$$

Thus :

$$\lambda\left(S(\tau)\right) = \frac{e_3 - e_1}{e_2 - e_1} = 1 - \lambda(\tau).$$

Notice that the last relation is coherent, since $\lambda(S^2(\tau)) = \lambda(\tau)$. You can see it by $S^2 = \mathrm{id}$, and $\lambda(S^2(\tau)) = 1 - \lambda(S(\tau)) = \lambda(\tau)$.
Compare with the function that we used during the proof of the little Picard theorem, where the function λ needed to satisfy some relation. In particular, you can compare with the compuations we did on the fundamental domain Q. Originally, this function was used by Picard to prove the eponymous theorem.

10.4 Application : elliptic integrals

10.4.1 The pendulum [12]

Let us recall first the non-linear equation satisfied by the angle θ of the simple pendulum problem:

$$ml\ddot{\theta} = -mg\sin(\theta).$$

m corresponds to the mass of a ball, g the gravity constant, and l the length of the wire. This equation is obtaind by the second Newton law. The angle θ is supposed, at the initial time $t = 0$, to be at its maximum $\theta_0 \in (0, \frac{\pi}{2})$, so the ball has maximal height. Since it is the maximum, we have $\dot{\theta}(0) = 0$. By multiplying the equation of the pendulum by $\dot{\theta}$, and by integrating from 0 to t :

$$l\frac{\dot{\theta}(t)^2}{2} - l\frac{\dot{\theta}(0)^2}{2} = g\left(\cos(\theta(t)) - \cos(\theta(0))\right),$$

Let us denote t_0 the first time at which $\theta(t_0) = 0$. By the initial condition $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, we get for any time $t \in (0, t_0)$:

$$\dot{\theta}(t) = -\sqrt{\frac{2g}{l}}\sqrt{\cos(\theta(t)) - \cos(\theta_0)}.$$

The sign is coherent in the previous equality, since the value of θ decays on $(0, t_0)$.
Let us now compute the period T (from θ_0 to θ_0 again):

$$\frac{T}{4} = -\sqrt{\frac{l}{2g}}\int_0^{\frac{T}{4}} \frac{\dot{\theta}(t)dt}{\sqrt{\cos(\theta(t)) - \cos(\theta_0)}} = -\frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\frac{T}{4}} \frac{\dot{\theta}(t)dt}{\sqrt{-\sin\left(\frac{\theta(t)}{2}\right)^2 + \sin\left(\frac{\theta_0}{2}\right)^2}}. \tag{17}$$

By the change of variable $k\sin(u) = \sin(\frac{\theta(t)}{2})$, where $k := \sin\left(\frac{\theta_0}{2}\right)$:

$$\begin{aligned} T &= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{-k^2\sin(u)^2 + k^2}} \frac{k\cos(u)}{\sqrt{1 - k^2\sin(u)^2}}du \quad \text{because } \frac{\dot{\theta}(t)}{2}dt = \frac{k\cos(u)}{\sqrt{1 - k^2\sin(u)^2}}du \\ &= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2\sin(u)^2}}. \end{aligned}$$

The last form of this integral is the usual definition of an elliptic integral of first kind, with a parameter $k \in (0, 1)$. Notice that when $k = 1$, we find the trigonometric function : it is not integrable at $\frac{\pi}{2}$ since $\frac{1}{\cos(\frac{\pi}{2} - h)} \sim \frac{1}{h}$.

To give the relation between this integral and elliptic functions, we perform the change of variable $\omega = -\cos(\theta(t))$ in the first integral of (17):

$$\begin{aligned}
T &= -4\sqrt{\frac{l}{2g}}\int_{-\cos(\theta_0)}^{-1} \frac{d\omega}{\sqrt{(1-\omega^2)(-\omega-\cos(\theta_0))}} \qquad \text{because } \dot{\theta}(t)dt = \frac{1}{\sqrt{1-\omega^2}}d\omega, \\
&= 4\sqrt{\frac{l}{2g}}\int_{-1}^{-\cos(\theta_0)} \frac{d\omega}{\sqrt{(\omega-1)(\omega+1)(\omega+\cos(\theta_0))}} \qquad \text{and with } z := \omega + \frac{\cos(\theta_0)}{3}, \\
&= 4\sqrt{\frac{2l}{g}}\int_{-1+\frac{1}{3}\cos(\theta_0)}^{-\frac{2}{3}\cos(\theta_0)} \frac{dz}{\sqrt{4\left(z-1-\frac{1}{3}\cos(\theta_0)\right)\left(z+1-\frac{1}{3}\cos(\theta_0)\right)\left(z+2\frac{\cos(\theta_0)}{3}\right)}}
\end{aligned}$$

In the previous factorization, we can see that the roots $1+\frac{1}{3}\cos(\theta_0)$, $-1+\frac{1}{3}\cos(\theta_0)$ and $-\frac{2}{3}\cos(\theta_0)$ are distinct since $\theta_0 \in (0, \frac{\pi}{2})$, and thus we can define the associated Weierstrass function. The last change of variable was necessary to obtain a polynomial of the form $4z^3 - g_2 z - g_3$. As with previous remark (168), this integral can be made explicit in term of the inverse function $\wp^{-1}$, along a path to define.

10.4.2 Some generalities on elliptic integrals.

There exist three kinds of elliptic integrals. We give their Legendre forms.

- Elliptic integral of first kind:

$$F(k,\Phi) := \int_0^{\Phi} \frac{d\theta}{\sqrt{1-k^2\sin(\theta)}}.$$

  One example is the previous computation on the pendulum. The function that we used was the $\wp$-function of Weierstrass, which is an elliptic function of order 2, with one pole of multiplicty 2.

- Elliptic integral of the second form:

$$E(k,\Phi) := \int_0^{\Phi} \sqrt{1-k^2\sin(\theta)}d\theta.$$

  The parameter k is in $(0;1)$. In fact, if $k=0$, we find the formula for the length of an arc of the disk. However, when $k \in (0,1)$, the formula corresponds (up to some constants) to the lenght of an arc of an ellipse.

  The relation with this integral with elliptic integrals is made via the Jacobi functions, which are elliptic function of order 2, with two simple poles.

- Elliptic integral of third kind:

$$H(k,n,\Phi) = \int_0^{\Phi} \frac{d\theta}{(1+n\sin(\theta)^2)\sqrt{1-k^2\sin(\theta)^2}}.$$

A general definition of an elliptic integral can be given as $\int R(x,y)dx$, with R a rational algebraic polynomial (id est : the quotient of two polynomials in x and in y). If $y = \sqrt{P(x)}$ and the denominator is a cubic or quartic polynomial, then $\int R(x, P(x))dx$ can be evaluated in terms of the elliptic integrals.
You can find more on elliptic integrals on [15] (a nice algebraic point of view).

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